

Asymptotic convergence to pushed wavefronts in a monostable equation with delayed reaction

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Abstract. We study the asymptotic behavior of solutions to the delayed monostable equation $(*)$: $u_t(t, x) = u_{xx}(t, x) - u(t, x) + g(u(t - h, x))$, $x \in \mathbb{R}$, $t > 0$, with monotone reaction term $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$. Our basic assumption is that this equation possesses pushed traveling fronts. First we prove that the pushed wavefronts are nonlinearly stable with asymptotic phase. Moreover, combinations of these waves attract, uniformly on \mathbb{R} , every solution of equation $(*)$ with the initial datum sufficiently rapidly decaying at one (or at the both) infinities of the real line. These results provide a sharp form of the theory of spreading speeds for equation $(*)$.

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1. Introduction and main results

In this work, we study the asymptotic convergence of solution $u(t, x)$ of the initial value problem for a monostable reaction-diffusion equation with delayed reaction

$$u_t(t, x) = u_{xx}(t, x) - u(t, x) + g(u(t - h, x)), \quad (1)$$

$$u(s, x) = w_0(s, x), \quad s \in [-h, 0], \quad x \in \mathbb{R}, \quad (2)$$

to a combination of traveling waves. In the sequel, it is always assumed that the continuous function $w_0(s, x)$ is locally Hölder continuous in $x \in \mathbb{R}$, uniformly with respect to s , and that the function $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ satisfies the monostability condition

(H) the equation $g(x) = x$ has exactly two nonnegative solutions: 0 and $\kappa > 0$. Moreover, g is C^1 -smooth in some δ_0 -neighborhood of the equilibria where $g'(0) > 1$, $g'(\kappa) < 1$, and also satisfies the Lipschitz condition $|g(u) - g(v)| \leq L_g|u - v|$, $u, v \in [0, \kappa]$. In addition, there are $C > 0$, $\theta \in (0, 1]$, such that $|g'(u) - g'(0)| + |g'(\kappa) - g'(\kappa - u)| \leq Cu^\theta$ for $u \in (0, \delta_0]$. Without restricting generality, we will also assume that g is linearly and C^1 -smoothly extended on $(-\infty, 0]$ and $[\kappa, +\infty)$.

Equation (1) (together with its non-local versions) is an important model in the population dynamics [6, 15, 17, 21, 23, 24, 27, 28, 45, 48, 49] where it is used to describe the spatio-temporal evolution of a single-species population. In this interpretation of (1), g is a birthrate function, $u(t, x)$ denotes the population density at location x and time t , and it is supposed that the species reaches sexual maturity at age $h > 0$. Clearly, the Cauchy problem (1), (2) can be solved by the method of steps [13], where in the first step we have to look for the solution of the inhomogeneous linear equation

$$u_t(t, x) = u_{xx}(t, x) - u(t, x) + g(w_0(t - h, x)), \quad t \in [0, h], \quad x \in \mathbb{R},$$

satisfying the initial condition $u(0, x) = w_0(0, x)$. Besides the hypothesis **(H)**, from a biological point of view, it is realistic to assume that the birth function g is either strictly increasing or unimodal (i.e. g has exactly one critical point which is the absolute maximum point [21, 41, 48]) function on \mathbb{R}_+ . In the population dynamics, equation (1) improves certain weaknesses (cf. [18] or [43, pp. 56-58]) of the logistic growth model given by the KPP-Fisher delayed or nonlocal equations [3, 5, 7, 9, 16, 20]. One of the most interesting features of the dynamics in (1) is the existence of smooth positive solutions $u(t, x) = \phi(x + ct)$ satisfying the boundary conditions $\phi(-\infty) = 0$ and $\liminf_{t \rightarrow +\infty} \phi(t) > 0$ (for $c > 0$, cf. [15]). Such solutions are called traveling semi-wavefronts (or wavefronts if additionally $\phi(+\infty) = \kappa$), they describe waves of colonisation propagating with the velocity c . The convergence and stability properties of wavefronts to (1) are quite well understood in the non-delayed case (i.e. when $h = 0$). The studies of the front stability in non-delayed monostable equation (1) were initiated in 1976 by Sattinger [36] (see [29] for the state-of-art on this topic), but already the seminal work of Kolmogorov, Petrovskii, Piskunov (1937) presented a first deep analysis of the convergence of the solution $u(t, x)$ of (1), (2) (with $-u + g(u) = u(1 - u)$ and with $w_0(s, x)$ being the Heaviside step function $H(x)$) to a monotone wavefront.

Now, the investigation of asymptotic behavior of solution to problem (1), (2) becomes a much more challenging task when $h > 0$. For instance, the recent works [7, 8, 15, 16, 20, 24, 41] show that the delay h has a strong influence on the geometry of front's profile ϕ and complicates enormously the studies of the front uniqueness [1, 6, 42, 45] and stability [6, 21, 23, 24, 25, 27, 28, 29, 45]. Moreover, in order to be able to perform the local stability analysis of equation (1), it was always necessary to assume the additional sub-tangency restriction

$$g(u) \leq g'(0)u, \quad u \geq 0. \quad (3)$$

Under this assumption, all wavefronts of equation (1) are known as '*pulled*' fronts (see [5, 14, 32, 33, 34, 39, 47] for further details), model (1) is linearly determined [19, 46] and there exists a positive number $c_* > 0$ (called the minimal speed of propagation) separating the positive axis on the set of admissible *semi-wavefronts* speeds $[c_*, +\infty)$ and the set $[0, c_*)$ of velocities c for which does not exist *any* non-constant positive bounded wave solution $u(t, x) = \phi(x + ct)$ [15]. Furthermore, the minimal speed c_* is determined from the characteristic equation

$$\chi(z, c) := z^2 - cz - 1 + g'(0)e^{-zch} = 0 \quad (4)$$

as the unique real value $c_\#$ for which $\chi(z, c)$ has a positive double zero $\lambda_1(c_\#) = \lambda_2(c_\#)$ (i.e. c_* is equal to $c_\#$ if (3) holds). Note that for $c > c_\#$ equation $\chi(z, c) = 0$ has exactly two positive simple roots, we will denote them as $\lambda_1(c) < \lambda_2(c)$.

In this way, as far as we know, all studies of wave's stability in the delayed model (1) have dealt exclusively with the stability of pulled wavefronts. Nevertheless, from an ecological point of view, models with the birth functions which are not sub-tangential at $u = 0$ are also quite interesting in view of the interpretation of non-sub-tangentiality property of g in terms of a weak Allee effect [5, 12, 32]. In the non-delayed case, it is well known [14, 32, 33, 34, 39, 47] that such systems can possess a special type of minimal wavefronts called the '*pushed*' fronts. As the characterising property of a pushed wave for model (1), we can take the following one: the minimal wavefront $u(t, x) = \phi(x + c_*t)$ is pushed if the velocity c_* is not linearly determined, i.e. if $c_* > c_\#$. The recent work [32] explains why, contrarily to the pulled waves, the pushed colonisation waves can be considered as waves promoting genetic diversity in the ecological systems.

To the best of our knowledge, the study of pushed waves in the monostable delayed model (1) was initiated in [23, 42] (curiously, in the first work [37] dealing with traveling waves in delayed models, all waves were tacitly presumed to be pulled). In [42], after assuming monotonicity of g , it was proved that the unique minimal wavefront propagating with the speed $c_* > c_\#$ must have a strictly increasing profile ϕ with the following asymptotic representation at $-\infty$:

$$\phi(t + s_0) = e^{\lambda_2 t} + O(e^{(\lambda_2 + \varsigma)t}), \quad \lambda_2 := \lambda_2(c_*), \quad \varsigma > 0, \quad t \rightarrow -\infty. \quad (5)$$

It should be noted that the situation when non-monotone (for example, unimodal) birth function $g : [0, \kappa] \rightarrow \mathbb{R}_+$ does not satisfy (3) is not completely understood till now. In fact, even the existence of the minimal speed of propagation c_* , as the lowest value from

a closed connected unbounded set of all admissible wavefront (or semi-wavefront [3, 15]) velocities, is not yet proved for the case of non-monotone and not sub-tangential g . From the formal point of view, the existence of the pushed fronts to the delayed model (1) neither was established in [42]. In any case, this point can be easily completed:

Proposition 1.1 *Assume that $u = \phi(x + c_*t)$, $c_* > c_\#(h_0)$, is a pushed traveling front to the monotone model (1) considered with some fixed $h_0 \geq 0$. Then there exists a positive δ such that equation (1) possesses a pushed traveling front for each non-negative $h \in (h_0 - \delta, h_0 + \delta)$. In particular, there exists a delayed equation (1) with $h > 0$ possessing the minimal monotone wavefront $u = \phi(x + c_*t)$ with the profile ϕ satisfying the asymptotic formula (5).*

Proof. Since $c_\#(h)$ depends continuously on $h \geq 0$, the first part of Proposition 1.1 will be proved if we establish the lower semicontinuity of $c_*(h)$ at h_0 . Then the existence of pushed wavefronts to the equation (1) considered with small positive delays follows from the existence of the pushed wavefronts to the Fisher type population genetic model [19, Theorem 11] $u_t(t, x) = u_{xx}(t, x) - u(t, x) + (10u(t, x) + 3u^2(t, x) - 5u^3(t, x))/8$. Hence, it suffices to prove the following

Claim. *Suppose that $h_j \rightarrow h_0$, $c_*(h_j) \rightarrow c_0$ as $j \rightarrow +\infty$. Then $c_0 \geq c_*(h_0)$.*

Indeed, take some $c > c_0$. Then, for all sufficiently large j , the equation

$$u_t(t, x) = u_{xx}(t, x) - u(t, x) + g(u(t - h_j, x))$$

has a unique (up to translation) positive strictly monotone wavefront $u(t, x) = \phi_j(x + ct)$. Without the loss of the generality, we can assume that $\phi_j(0) = \kappa/2$. It is easy to see (cf. [42]) that each profile ϕ_j satisfies the integral equation

$$\phi(t) = \frac{1}{\xi_2 - \xi_1} \left(\int_{-\infty}^t e^{\xi_1(t-s)} g(\phi(s - ch_j)) ds + \int_t^{+\infty} e^{\xi_2(t-s)} g(\phi(s - ch_j)) ds \right), \quad (6)$$

where $\xi_1 < 0 < \xi_2$ are roots of the equation $z^2 - cz - 1 = 0$. Since $|\phi'_j(t)| \leq \kappa/\sqrt{c^2 + 4}$, $|\phi_j(t)| \leq \kappa$, the sequence ϕ_j has a subsequence ϕ_{j_k} which converges, uniformly on compact subsets of \mathbb{R} , to the monotone continuous bounded function $\phi_0(t)$, $\phi_0(0) = \kappa/2$. By the Lebesgues dominated convergence theorem, ϕ_0 satisfies the equation (6) with h_0 and therefore ϕ_0 is a positive profile of strictly monotone wavefront propagating with the velocity c [15, 42]. In this way, $c \geq c_*(h_0)$ for every $c > c_0$ that yields $c_0 \geq c_*(h_0)$. ■

Formula (5) implies that pushed profiles $\phi(s)$ converges to 0 at $-\infty$ more rapidly than the profiles of other (i.e. non-minimal or pulled) waves behaving as

$$\phi(t + s_0) = (-t)^m e^{\lambda_1 t} + O(e^{(\lambda_1 + \varsigma)t}), \quad \lambda_1 := \lambda_1(c), \quad \varsigma > 0, \quad m \in \{0, 1\}, \quad t \rightarrow -\infty.$$

The fast asymptotic decay of pushed fronts at $-\infty$ makes them similar to the so-called *bistable* fronts [10, 38, 47]. Actually, by analysing the inside dynamics of wavefronts, Garnier *et al* [14] (in the non-delayed case) and Bonnefon *et al* [5] (in the delayed case) have recently proposed a general definition of pushed waves which allows to consider

the monostable pushed fronts and the bistable fronts within a unified framework. An additional argument in favor of this insight is provided by the theory of nonlinear stability of waves. Indeed, both monostable pushed fronts and bistable fronts are proved to have rather good stability properties [10, 30, 33, 38, 39, 40]. Furthermore, the most complete and comprehensible proof of the asymptotic stability of monostable pushed front given in [33] uses constructions and results obtained for a bistable model in [10].

Hence, the main aim of the present paper is to study the stability properties of monostable pushed fronts to the monotone delayed model (1). We are going to achieve this goal by developing several ideas and methods from [10, 30, 33, 42]. We also will establish the asymptotic convergence of solutions for the initial value problem (1), (2) to an appropriate pushed wavefront when, in addition to (H), g is monotone and when w_0 satisfies, for some $A, B > 0$, $\sigma \in (0, \kappa)$ and $\mu > \lambda_1(c_*)$ the following conditions (IC):

$$(IC1) \quad 0 \leq w_0(s, x) \leq \kappa, \quad x \in \mathbb{R}, \quad s \in [-h, 0];$$

$$(IC2) \quad w_0(s, x) \leq Ae^{\mu x}, \quad x \in \mathbb{R}, \quad s \in [-h, 0];$$

$$(IC3) \quad w_0(s, x) > \kappa - \sigma, \quad s \in [-h, 0], \quad x \geq B.$$

From the monotonicity of g and the hypotheses (H), (IC), by invoking the well-known existence and uniqueness results and the comparison principle [11, Chapter 1, Theorems 12, 16], we can deduce the existence of a unique classical solution $u = u(t, x) : [-h, +\infty) \times \mathbb{R} \rightarrow [0, \kappa]$ to (1), (2) (i.e. of a continuous bounded function u having continuous derivatives u_t, u_x, u_{xx} in $\Omega = (0, +\infty) \times \mathbb{R}$ and satisfying (1) in Ω as well as (2) in $[-h, 0] \times \mathbb{R}$). As the following proposition shows, the asymptotic behavior of this solution $u(t, x)$ on bounded subsets of \mathbb{R} is quite simple:

Proposition 1.2 *Suppose that the initial datum $w_0 \not\equiv 0$ satisfies (IC1) and that the Lipschitz continuous map $g : [0, \kappa] \rightarrow [0, \kappa]$ has exactly two fixed points: 0 and $\kappa > 0$. Then $\lim_{t \rightarrow \infty} u(t, x) = \kappa$ uniformly on compact subsets of \mathbb{R} .*

At first glance, if additionally we assume the monotonicity of g , Proposition 1.2 seems to follow from quite general results on spreading speeds to continuous-time semiflows established in [22, 23]. Indeed, [23, Theorem 34] shows that even rather weak positivity condition assumed in Proposition 1.2 is enough to assure stronger convergence

$$\lim_{t \rightarrow \infty} \sup_{x \in [-c't, c't]} |u(t, x) - \kappa| = 0, \quad c' \in (0, c_*), \quad (7)$$

once g is a subhomogeneous function: $\rho g(x) \leq g(\rho x)$ for all $\rho \in [0, 1]$ and $x \geq 0$. It is easy to see, however, that the latter condition implies the sub-tangency inequality (3).

Our proof of Proposition 1.2 follows closely the main lines of [2], where Aronson and Weinberger established a similar result for non-delayed equations. See also [49, Theorem 3.2] for an analogous assertion proved for a non-diffusive delay differential equation with spatial non-locality in an unbounded domain. In general (e.g. under condition (IC2)) the convergence of $u(t, \cdot) \rightarrow \kappa$, $t \rightarrow +\infty$, is not uniform on \mathbb{R} : this is an immediate outcome of our subsequent investigation of the asymptotic behavior of the entire solution $u(t, x)$ as $t \rightarrow +\infty$ on the whole real x -line \mathbb{R} .

In order to state the main results of this paper, we take a pushed front $\phi(x + c_*t)$ for equation (1) and fix a positive number $\lambda < \mu$ such that $\lambda \in (\lambda_1(c_*), \lambda_2(c_*))$. We will also consider the Banach space

$$C_\lambda(\mathbb{R}) = \left\{ y \in C(\mathbb{R}, \mathbb{R}) : |y|_\lambda := \max\left\{\sup_{x \leq 0} e^{-\lambda x} |y(x)|, \sup_{x \geq 0} |y(x)|\right\} < \infty \right\}.$$

Observe that $|y|_\lambda = \sup_{x \in \mathbb{R}} |y(x)|/\eta(x)$, where $\eta(x) := \min\{e^{\lambda x}, 1\}$. Our first theorem shows that the pushed front $\phi(x + c_*t)$, $c_* > c_\#$, is nonlinearly stable with asymptotic phase [35]:

Theorem 1.3 *Let g be monotone and conditions (IC), (H) be satisfied. Then for every $\epsilon > 0$ there exists $\delta > 0$ such that $|\phi(\cdot + c_*s) - w_0(s, \cdot)|_\lambda < \delta$, $s \in [-h, 0]$, implies that $|\phi(\cdot + c_*t) - u(t, \cdot)|_\lambda < \epsilon$ for all $t \geq 0$. Here $u(t, x)$ is solution of the initial value problem (1), (2). Furthermore, there exists s_0 such that $|\phi(\cdot + c_*t + s_0) - u(t, \cdot)|_\lambda \rightarrow 0$ as $t \rightarrow +\infty$.*

The stability result of Theorem 1.3 follows from Corollary 2.4 proved in Section 2 while the asymptotic convergence $u(t, x) \rightarrow \phi(x + c_*t + s_0)$, $t \rightarrow +\infty$, follows from the next theorem. It describes the global stability properties of the pushed fronts with respect to initial data satisfying the hypothesis (IC):

Theorem 1.4 *Let g be monotone and conditions (IC), (H) be satisfied. Then the solution $u(t, x)$ of the initial value problem (1), (2) asymptotically converges to a shifted front. In fact, for some $s_0 \in \mathbb{R}$,*

$$\lim_{t \rightarrow \infty} \sup_{x \in \mathbb{R}} |u(t, x) - \phi(x + c_*t + s_0)|/\eta(x + c_*t) = 0. \quad (8)$$

It is instructive to compare Theorems 1.3, 1.4 with stability results obtained for non-critical pulled fronts in the delayed model (1) with monotone reaction g satisfying (3) and (H). For example, taking initial functions w_0 satisfying (IC1) and assuming that the initial disturbance $\phi(\cdot + cs) - w_0(s, \cdot)$ belongs to the weighted Sobolev space $H_{\eta^2}^1(\mathbb{R})$ and depends continuously on $s \in [-h, 0]$, Mei et al [27, Theorem 2.2] proved that $|\phi(\cdot + ct) - u(t, \cdot)|_0 \rightarrow 0$ exponentially when $t \rightarrow +\infty$. Hence, in view of the continuous imbedding $H_{\eta^2}^1(\mathbb{R}) \subset C_\lambda(\mathbb{R}) \cap C^{0,1/2}(\mathbb{R}_+)$, initial functions $w_0(s, x)$ in [27] are uniformly Hölder continuous in x and converge at $+\infty$, $w_0(s, +\infty) = \kappa$ (in fact, this convergence is uniform in $s \in [-h, 0]$, so that each w_0 meets trivially the restriction (IC3)). They should also satisfy the inequality

$$|\phi(x + cs) - w_0(s, x)| \leq Ce^{\lambda x}, \quad x \in \mathbb{R}, \quad s \in [-h, 0], \quad \text{for some } C > 0, \quad \lambda \in (\lambda_1, \lambda_2). \quad (9)$$

Due to the asymptotic representation (5) and to certain freedom in the choice of λ, μ , in the case of pushed fronts, the latter condition amounts precisely to the hypothesis (IC2). Nevertheless, in contrast to inequality (9) considered with a pushed front $u = \phi(x + c_*s)$, the same inequality considered with a pulled front $u = \phi(x + cs)$ is not satisfied if we take the Heaviside step function $H(x)$ as the initial function $w_0(s, x) = H(x)$. Thus the question about the asymptotic form of solution $u(t, x)$ to the Cauchy problem (1), (2) with $w_0(s, x) = H(x)$ and with the sub-tangential g still remains unanswered in the

delayed case. It is worth to recall that precisely this question formulated for a non-delayed monostable equation (1) was the main object of studies in the seminal work by Kolmogorov, Petrovskii, Piskunov in 1937.

Now, it is worth noticing that equation (1) is invariant with respect to the transformation $x \rightarrow -x$ so that the statements of Theorems 1.3 and 1.4 can be easily adapted to the case when the initial function $w_0(s, -x)$ meets the hypothesis (IC). Evidently, in such a case, we should use *pushed backs* of the form $u = \phi(-x + c_*t)$ instead of the pushed wavefronts. Then the natural question is whether solution $u(t, x)$ converges to a combination of a pushed front and a pushed back when the both non-zero functions $w_0(s, x), w_0(s, -x)$ satisfy conditions (IC1), (IC2). In particular, this happens when w_0 has compact support. To the best of our knowledge, the studies of the asymptotic form of solutions to the monostable reaction-diffusion equations having compactly supported initial data were initiated in [2, 33, 40, 44]. Here, we analyse a similar problem in the presence of delay; hence, our third theorem considers the initial data for (1), (2) exponentially vanishing at both infinities.

Theorem 1.5 *Assume that $u = \phi(x + c_*t)$, $c_* > c_\#$, is a pushed traveling front to equation (1). If non-zero functions $w_0(s, x), w_0(s, -x)$ satisfy conditions (IC1), (IC2) then the solution $u = u(t, x)$ of the initial value problem (1), (2) asymptotically converges to a combination of two shifted fronts, i.e. for some $s_1, s_2 \in \mathbb{R}$,*

$$\begin{aligned} \lim_{t \rightarrow \infty} \sup_{x \leq 0} |u(t, x) - \phi(x + c_*t + s_1)| / \eta(x + c_*t) &= 0, \\ \lim_{t \rightarrow \infty} \sup_{x \geq 0} |u(t, x) - \phi(-x + c_*t + s_2)| / \eta(-x + c_*t) &= 0. \end{aligned}$$

Clearly, Theorem 1.5 combined with the comparison principle shows that relation (7) holds for each solution $u = u(t, x)$ to (1) once associated initial datum $w_0(s, x) \not\equiv 0$ satisfies (IC1). Moreover, since Theorem 1.5 implies that

$$\lim_{t \rightarrow \infty} \sup_{x \notin (-c't, c't)} u(t, x) = 0, \quad c' > c_*,$$

we can conclude that the speed c_* of pushed waves coincides with the spreading speed for model (1). Without restriction (3), this important result was for the first time established in [22, 23] (in a much more general setting). Therefore Theorem 1.5 can be also viewed as an essential improvement of the mentioned Liang and Zhao result for the particular case of Eq. (1).

As in [10, 33], the method of sub- and super-solutions is a key tool for proving our main results. The sub- and super-solutions will be obtained as suitable deformations (invented by Fife and McLeod in [10] for the bistable systems and adapted by Rothe in [33] for the monostable equations) of the pushed wavefront. The other important idea exploited in [10, 33] is the use of an appropriate Lyapunov functional for proving the wave stability. However, the construction of such a functional seems to be a rather difficult task in the case of the functional differential equation (1). Thus, instead of this, we decided to use the Berestycki and Nirenberg method of the sliding solutions

[4, 42] as well as some ideas of the approach developed by Ogiwara and Matano in [30]. It is natural to expect that the rate of convergence in (8) is exponential, see e.g. [10, 27, 28, 33, 35]. The demonstration of this fact, however, is based on a different approach and will be considered in a separate work.

Finally, we say a few words about the organization of the paper. The results of Theorems 1.3 and 1.4 follow from Corollary 2.4 and Theorem 2.10 which are proved in Section 2. Then various auxiliary results are proved in Section 3 (Proposition 1.2) and Section 4 (an important stability Lemma 4.4 among others). In the last section of the paper, we complete the proof of Theorem 1.5.

2. Proof of Theorems 1.3 and 1.4

Let $u = \phi(x + c_*t)$, $c_* > c_\#$, be a pushed traveling front to equation (1). In the sequel, to simplify the notation, we will avoid the subscript $*$ in c_* so that $u = \phi(x + c_*t) = \phi(x + ct)$. As it is usual, we consider the moving coordinate frame (t, z) where $z = x + ct$. Set $w(t, z) = u(t, z - ct)$, then equation (1) takes the form

$$w_t(t, z) = w_{zz}(t, z) - cw_z(t, z) - w(t, z) + g(w(t - h, z - ch)), \quad (10)$$

$$w(s, z) = \tilde{w}_0(s, z) := w_0(s, z - cs), \quad s \in [-h, 0], \quad z \in \mathbb{R}. \quad (11)$$

First, following Fife and McLeod [10, Lemma 4.1] and Rothe [33, Lemma 1], we prove the next assertion.

Lemma 2.1 *Assume that the hypothesis (H) is satisfied. Then there exist positive constants γ, C, q_0^+ (depending only on g, ϕ, c, h, λ) and $q_0^- = \sigma$ such that the inequality*

$$0 \leq w(s, z) \leq \phi(z) + q\eta(z), \quad z \in \mathbb{R}, \quad s \in [-h, 0], \quad (12)$$

with $q \in (0, q_0^+]$ implies

$$0 \leq w(t, z) \leq \phi(z + Cq) + qe^{-\gamma t}\eta(z), \quad z \in \mathbb{R}, \quad t \geq -h. \quad (13)$$

Similarly, the inequality

$$\phi(z + Cq) - q\eta(z) \leq w(s, z) \leq \kappa, \quad z \in \mathbb{R}, \quad s \in [-h, 0], \quad (14)$$

with $q \in (0, q_0^-]$ implies

$$\phi(z) - qe^{-\gamma t}\eta(z) \leq w(t, z) \leq \kappa, \quad z \in \mathbb{R}, \quad t \geq -h. \quad (15)$$

Proof. For the convenience of the reader, the proof is divided into five steps. Recall that the positive numbers δ_0, σ are defined in (H) and (IC), respectively.

Step I. We claim that given $\sigma \in (0, \kappa)$, there are positive $\delta_1^* < \delta_0$, $\gamma_1^* < \lambda c$ such that

$$g(u) - g(u - qe^{\gamma h}) \leq q(1 - 2\gamma), \quad \text{for all } (u, q) \in [\kappa - \delta_1^*, \kappa + \delta_1^*] \times [0, \sigma], \quad \gamma \in [0, \gamma_1^*].$$

Indeed, it suffices to note that, given $\sigma \in (0, \kappa)$, the continuous function

$$G(u, q, \gamma) := \begin{cases} 1 + (g(u - e^{\gamma h}q) - g(u))/q, & (u, q) \in [\kappa - \delta_1^*, \kappa + \delta_1^*] \times (0, \sigma], \quad \gamma \in [0, \gamma_1^*]; \\ 1 - e^{\gamma h}g'(u), & u \in [\kappa - \delta_1^*, \kappa + \delta_1^*], \quad q = 0, \quad \gamma \in [0, \gamma_1^*], \end{cases}$$

satisfies $G(\kappa, q, 0) > 2\gamma_1^*$, $q \in [0, \sigma]$, for sufficiently small γ_1^*, δ_1^* (recall that $g'(\kappa) < 1$). Thus $G(u, q, \gamma) > 2\gamma$ for all $(u, q) \in [\kappa - \delta_1^*, \kappa + \delta_1^*] \times [0, \sigma]$, $\gamma \in [0, \gamma_1^*]$ if γ_1^*, δ_1^* are sufficiently small.

Step II. As in [10, 33], we have to construct appropriate super- and sub-solutions. Consider the nonlinear operator \mathcal{N} defined as

$$\mathcal{N}w(t, z) := w_t(t, z) - w_{zz}(t, z) + cw_z(t, z) + w(t, z) - g(w(t - h, z - ch)).$$

By definition, continuous function $w_+ : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$ is called a super-solution for (10), if, for some $z_* \in \mathbb{R}$, this function is $C^{1,2}$ -smooth in the domains $\mathbb{R}_+ \times (-\infty, z_*]$ and $\mathbb{R}_+ \times [z_*, +\infty)$ and

$$\mathcal{N}w_+(t, z) \geq 0 \text{ for } t > 0, z \neq z_*, \text{ while } (w_+)_z(t, z_*-) \geq (w_+)_z(t, z_*+) \text{ for } t > 0. \quad (16)$$

Sub-solutions w_- are defined analogously, with the inequalities " \geq " reversed in (16). We will look for super- and sub-solutions of the form

$$w_+(t, z) := \phi(z + \epsilon(t)) + qe^{-\gamma t}\eta(z), \quad w_-(t, z) := \phi(z - \epsilon_1(t)) - qe^{-\gamma t}\eta(z),$$

where, for appropriate positive parameters α, γ (to be fixed later and depending only on g, ϕ, c, h, λ), increasing $\epsilon(t)$, $\epsilon_1(t)$ are defined by

$$\epsilon(t) := \frac{\alpha q}{\gamma}(e^{\gamma h} - e^{-\gamma t}) > 0, \quad \epsilon_1(t) := \epsilon(t) - \epsilon(+\infty) = -\frac{\alpha q}{\gamma}e^{-\gamma t} < 0, \quad t > -h.$$

Note that the smoothness conditions and the second inequality in (16) with $z_* = 0$ are obviously fulfilled because of

$$\frac{\partial w_+(t, 0+)}{\partial z} - \frac{\partial w_+(t, 0-)}{\partial z} = -q\lambda e^{-\gamma t} < 0, \quad \frac{\partial w_-(t, 0+)}{\partial z} - \frac{\partial w_-(t, 0-)}{\partial z} = q\lambda e^{-\gamma t} > 0,$$

so that we have to check the first inequality of (16) only. Since g, ϕ, ϵ are strictly increasing, we have, for $z \neq 0$, that

$$\begin{aligned} \mathcal{N}w_+(t, z) &:= \epsilon'(t)\phi'(z + \epsilon(t)) - \gamma qe^{-\gamma t}\eta(z) - \phi''(z + \epsilon(t)) - qe^{-\gamma t}\eta''(z) + c\phi'(z + \epsilon(t)) \\ &+ cq e^{-\gamma t}\eta'(z) + \phi(z + \epsilon(t)) + qe^{-\gamma t}\eta(z) - g(w_+(t - h, z - ch)) \geq \alpha qe^{-\gamma t}\phi'(z + \epsilon(t)) \\ &- \gamma qe^{-\gamma t}\eta(z) + cq e^{-\gamma t}\eta'(z) + qe^{-\gamma t}\eta(z) - qe^{-\gamma t}\eta''(z) \\ &+ g(\phi(z - ch + \epsilon(t))) - g(\phi(z - ch + \epsilon(t)) + qe^{-\gamma(t-h)}\eta(z - ch)); \end{aligned}$$

$$\begin{aligned} \mathcal{N}w_-(t, z) &:= -\epsilon_1'(t)\phi'(z - \epsilon_1(t)) + \gamma qe^{-\gamma t}\eta(z) - \phi''(z - \epsilon_1(t)) + qe^{-\gamma t}\eta''(z) + c\phi'(z - \epsilon_1(t)) \\ &- cq e^{-\gamma t}\eta'(z) + \phi(z - \epsilon_1(t)) - qe^{-\gamma t}\eta(z) - g(w_-(t - h, z - ch)) \leq -\alpha qe^{-\gamma t}\phi'(z - \epsilon_1(t)) \\ &+ \gamma qe^{-\gamma t}\eta(z) - cq e^{-\gamma t}\eta'(z) - qe^{-\gamma t}\eta(z) + qe^{-\gamma t}\eta''(z) + g(\phi(z - ch - \epsilon_1(t))) \\ &- g(\phi(z - ch - \epsilon_1(t)) - qe^{-\gamma(t-h)}\eta(z - ch)). \end{aligned}$$

Since $\lambda \in (\lambda_1(c), \lambda_2(c))$ and $g'(0) > 1$, we can choose sufficiently small $\gamma \in (0, \gamma_1^*)$ and $\delta \in (0, \kappa/2) \cap (0, \delta_1^*) \cap (0, \sigma)$, such that, for all $\bar{s} < \delta$ it holds

$$-\lambda^2 + c\lambda + 1 - \gamma - g'(\bar{s})e^{-\lambda ch + \gamma h} > 0. \quad (17)$$

In addition, we can take δ such that the unique real roots $z_0 < z_1 < z_2$ of the equations

$$\phi(z_0) = \delta/4, \quad \phi(z_1) + 0.25\delta\eta(z_1) = \delta/2; \quad \phi(z_2) = \kappa - \delta/2,$$

are such that $z_1 < -ch < 0 < z_2$. From now on, we will fix α, q_0^\pm defined by

$$q_0^+ = \delta e^{-\gamma h}/2, \quad q_0^- = \sigma, \quad \alpha = (\gamma + e^{\gamma h} L_g)/\beta, \quad \text{with } \beta := \min_{z \in [z_0, z_2 + ch]} \phi'(z).$$

We observe that α, q_0^\pm and γ depends only on $g, \phi, c, h, \lambda, \sigma$.

Step III. We claim that $\mathcal{N}w_+(t, z) \geq 0$ for all $z \neq 0$, $t \geq 0$ and $q \leq q_0^+ = \delta e^{-\gamma h}/2$.

Indeed, suppose first that $z - ch + \epsilon(t) \leq z_1$, then $z \leq z_1 + ch - \epsilon(t) < -\epsilon(t) < 0$ and

$$\phi(z - ch + \epsilon(t)) < \delta/2, \quad \phi(z - ch + \epsilon(t)) + qe^{-\gamma(t-h)}\eta(z - ch) < \delta.$$

As a consequence, we can invoke the mean value theorem and (17) to conclude that, for some $\bar{s} \in (0, \delta)$,

$$\begin{aligned} g(\phi(z - ch + \epsilon(t))) - g(\phi(z - ch + \epsilon(t)) + qe^{-\gamma(t-h)}\eta(z - ch)) &= -qe^{-\gamma(t-h)}\eta(z - ch)g'(\bar{s}), \\ \mathcal{N}w_+(t, z) &\geq -\gamma qe^{-\gamma t}\eta(z) + cqe^{-\gamma t}\eta'(z) + qe^{-\gamma t}\eta(z) - qe^{-\gamma t}\eta''(z) - qe^{-\gamma(t-h)}\eta(z - ch)g'(\bar{s}) \\ &= qe^{-\gamma t}([1 - \gamma]\eta(z) + c\eta'(z) - \eta''(z) - e^{\gamma h}\eta(z - ch)g'(\bar{s})) \\ &= qe^{-\gamma t + \lambda z}(1 - \gamma + c\lambda - \lambda^2 - e^{\gamma h - \lambda ch}g'(\bar{s})) > 0. \end{aligned}$$

Similarly, if $z - ch + \epsilon(t) \geq z_2$, then we have that

$$\kappa + \delta/2 \geq \phi(z - ch + \epsilon(t)) + qe^{-\gamma(t-h)}\eta(z - ch) \geq \phi(z - ch + \epsilon(t)) \geq \kappa - \delta/2.$$

Therefore, due to Step I and (17), for all $t \geq 0$,

$$g(\phi(z - ch + \epsilon(t))) - g(\phi(z - ch + \epsilon(t)) + qe^{-\gamma(t-h)}\eta(z - ch)) \geq -qe^{-\gamma t}\eta(z - ch)(1 - 2\gamma),$$

$$\begin{aligned} \mathcal{N}w_+(t, z) &\geq qe^{-\gamma t}([1 - \gamma]\eta(z) + c\eta'(z) - \eta''(z) - (1 - 2\gamma)\eta(z - ch)) \geq \\ &qe^{-\gamma t} \begin{cases} e^{\lambda z}[1 - \gamma + c\lambda - \lambda^2 - e^{-\lambda ch}(1 - 2\gamma)], & z < 0 \\ \gamma, & z > 0 \end{cases} > 0. \end{aligned}$$

Finally, if $z_1 < z - ch + \epsilon(t) < z_2$, we find that

$$\phi(z - ch + \epsilon(t)) < \kappa - \delta/2, \quad \phi(z - ch + \epsilon(t)) + 0.25\delta\eta(z - ch + \epsilon(t)) > \delta/2$$

so that $\phi(z - ch + \epsilon(t)) > \delta/2 - 0.25\delta\eta(z - ch + \epsilon(t)) > \delta/4$, and $z + \epsilon(t) \in [z_0, z_2 + ch]$. Obviously,

$$|g(\phi(z - ch + \epsilon(t))) - g(\phi(z - ch + \epsilon(t)) + qe^{-\gamma(t-h)}\eta(z - ch))| \leq L_g qe^{-\gamma(t-h)}\eta(z - ch).$$

Therefore, since $\eta(z) + c\eta'(z) - \eta''(z) > 0$ for $z \neq 0$ and $\eta(z) \in (0, 1]$, we get

$$\mathcal{N}w_+(t, z) \geq qe^{-\gamma t} \{ \alpha\beta + \eta(z) + c\eta'(z) - \eta''(z) - \gamma - e^{\gamma h}L_g \} > 0$$

for all $t \geq 0$. Hence, there exist some constants $\alpha, \gamma, q_0^+ > 0$, depending only on the wavefront profile ϕ , the nonlinearity g and c, h, λ such that, for any choice of $q \in (0, q_0^+)$ it holds $\mathcal{N}w_+(t, z) > 0$ for all $t \geq 0$ and $z \neq 0$. This proves the first inequality in (16).

Step IV. We claim that $\mathcal{N}w_-(t, z) \leq 0$ for all $z \neq 0$, $t \geq 0$ and $q \leq q_0^- = \sigma$.

Indeed, suppose first that $z - ch - \epsilon_1(t) \leq z_1$, then $z \leq z_1 + \epsilon_1(t) + ch < 0$ and

$$\phi(z - ch - \epsilon_1(t)) - qe^{-\gamma(t-h)}\eta(z - ch) < \phi(z - ch - \epsilon_1(t)) < \delta/2.$$

As a consequence, the mean value theorem yields that for some $\bar{s} < \delta$,

$$\begin{aligned} g(\phi(z - ch - \epsilon_1(t))) - g(\phi(z - ch - \epsilon_1(t)) - qe^{-\gamma(t-h)}\eta(z - ch)) &= qe^{-\gamma(t-h)}\eta(z - ch)g'(\bar{s}), \\ \mathcal{N}w_-(t, z) &\leq \gamma qe^{-\gamma t}\eta(z) - cq e^{-\gamma t}\eta'(z) - qe^{-\gamma t}\eta(z) + qe^{-\gamma t}\eta''(z) + qe^{-\gamma(t-h)}\eta(z - ch)g'(\bar{s}) \\ &= -qe^{-\gamma t}([1 - \gamma]\eta(z) + c\eta'(z) - \eta''(z) - e^{\gamma h}\eta(z - ch)g'(\bar{s})) \\ &= -qe^{-\gamma t}e^{\lambda z}(1 - \gamma + c\lambda - \lambda^2 - e^{\gamma h - \lambda ch}g'(\bar{s})) < 0. \end{aligned}$$

Similarly, if $z - ch - \epsilon_1(t) \geq z_2$, then we have that $\phi(z - ch - \epsilon_1(t)) \geq \kappa - \delta/2$ and therefore

$$g(\phi(z - ch - \epsilon_1(t))) - g(\phi(z - ch - \epsilon_1(t)) - qe^{-\gamma(t-h)}\eta(z - ch)) \leq (1 - 2\gamma)qe^{-\gamma t}\eta(z - ch)$$

for all $t \geq 0$, $q \in [0, \sigma]$. In consequence,

$$\begin{aligned} \mathcal{N}w_-(t, z) &\leq -qe^{-\gamma t}([1 - \gamma]\eta(z) + c\eta'(z) - \eta''(z) - (1 - 2\gamma)\eta(z - ch)) \leq \\ &\quad - qe^{-\gamma t} \begin{cases} e^{\lambda z}[1 - \gamma + c\lambda - \lambda^2 - e^{-\lambda ch}(1 - 2\gamma)], & z < 0 \\ \gamma, & z > 0 \end{cases} < 0. \end{aligned}$$

Finally, if $z_1 < z - ch - \epsilon_1(t) < z_2$, we find that

$$\phi(z - ch - \epsilon_1(t)) < \kappa - \delta/2, \quad \phi(z - ch - \epsilon_1(t)) + 0.25\delta\eta(z - ch - \epsilon_1(t)) > \delta/2$$

so that $\phi(z - ch - \epsilon_1(t)) > \delta/2 - 0.25\delta\eta(z - ch - \epsilon_1(t)) > \delta/4$, and $z - \epsilon_1(t) \in [z_0, z_2 + ch]$. Obviously,

$$|g(\phi(z - ch - \epsilon_1(t))) - g(\phi(z - ch - \epsilon_1(t)) - qe^{-\gamma(t-h)}\eta(z - ch))| \leq L_g qe^{-\gamma(t-h)}\eta(z - ch).$$

Therefore, since $\eta(z) + c\eta'(z) - \eta''(z) > 0$ for $z \neq 0$ and $\eta(z) \in (0, 1]$, we get

$$\mathcal{N}w_-(t, z) \leq -qe^{-\gamma t}\{\alpha\beta + \eta(z) + c\eta'(z) - \eta''(z) - \gamma - e^{\gamma h}L_g\} < -qe^{-\gamma t}(\gamma + L_g) < 0$$

for $t \geq 0$.

Step V. In view of (12) and the monotonicity properties of g , we have that

$$g(w_+(t - h, z - ch)) - g(w(t - h, z - ch)) \geq 0, \quad t \in [0, h], \quad z \in \mathbb{R}.$$

Therefore the difference $\delta(t, z) := w(t, z) - w_+(t, z)$ satisfies the inequalities

$$\begin{aligned} \delta(0, z) &\leq 0, \quad |\delta(t, z)| \leq \kappa + q_0^+, \quad \delta_{zz}(t, z) - \delta_t(t, z) - c\delta_z(t, z) - \delta(t, z) = \\ \mathcal{N}w_+(t, z) - \mathcal{N}w(t, z) + g(w_+(t - h, z - ch)) - g(w(t - h, z - ch)) &= \\ \mathcal{N}w_+(t, z) + g(w_+(t - h, z - ch)) - g(w(t - h, z - ch)) &\geq 0, \quad t \in (0, h], \quad z \in \mathbb{R}, \quad z \neq 0; \\ \frac{\partial \delta(t, 0+)}{\partial z} - \frac{\partial \delta(t, 0-)}{\partial z} &= q\lambda e^{-\gamma t} > 0, \quad t \in (0, h]. \end{aligned} \tag{18}$$

We claim that $\delta(t, z) \leq 0$ for all $t \in [0, h]$, $z \in \mathbb{R}$. Indeed, otherwise there exists $r_0 > 0$ such that $\delta(t, z)$ restricted to any rectangle $\Pi_r = [-r, r] \times [0, h]$ with $r > r_0$, reaches its maximal positive value $M > 0$ at at some point $(t', z') \in \Pi_r$.

We claim that (t', z') belongs to the parabolic boundary $\partial\Pi_r$ of Π_r . Indeed, suppose on the contrary, that $\delta(t, z)$ reaches its maximal positive value at some point (t', z') of $\Pi_r \setminus \partial\Pi_r$. Then clearly $z' \neq 0$ because of (18). Suppose, for instance that $z' > 0$. Then $\delta(t, z)$ considered on the subrectangle $\Pi = [0, r] \times [0, h]$ reaches its maximal positive value M at the point $(t', z') \in \Pi \setminus \partial\Pi$. Then the classical results [31, Chapter 3, Theorems 5,7] shows that $\delta(t, z) \equiv M > 0$ in Π , a contradiction.

Hence, the usual maximum principle holds for each Π_r , $r \geq r_0$, so that we can appeal to the proof of the Phragmén-Lindelöf principle from [31] (see Theorem 10 in Chapter 3 of this book), in order to conclude that $\delta(t, z) \leq 0$ for all $t \in [0, h]$, $z \in \mathbb{R}$.

But then we can again repeat the above argument on the intervals $[h, 2h]$, $[2h, 3h]$, \dots establishing that the inequality

$$0 \leq w(s, z) \leq \phi(z + \epsilon(s)) + qe^{-\gamma s}\eta(z), \quad z \in \mathbb{R},$$

actually holds for all $s \geq -h$. Since $\epsilon(t)$ increases on \mathbb{R} , this proves (13) with $C = \epsilon(\infty) = \alpha e^{\gamma h}/\gamma$.

Since the same method applied (with $C = \alpha e^{\gamma h}/\gamma$ in (14)) to the difference $\delta_-(t, z) := w_-(t, z) - w(t, z)$ leads to

$$\phi(z) - qe^{-\gamma s}\eta(z) < \phi(z - \epsilon_1(s)) - qe^{-\gamma s}\eta(z) \leq w(t, z) \leq \kappa, \quad t \geq -h, \quad z \in \mathbb{R},$$

the proof of the lemma is completed. \blacksquare

Remark 2.2 *It is worthwhile to note that the constants γ, C, q_0^\pm depend only on the form of ϕ in the sense that they will not change if we replace $\phi(z)$ with a shifted profile $\phi(z + b)$, $b \in \mathbb{R}$, in the statement of Lema 2.1.*

Due to Remark 2.2, the inequalities (14), (15) can be presented in the form similar to (12), (13):

Corollary 2.3 *Assume that the hypothesis (H) is satisfied. Then the inequality*

$$\phi(z) - q\eta(z) \leq w(s, z) \leq \kappa, \quad z \in \mathbb{R}, \quad s \in [-h, 0],$$

with $q \in (0, \sigma]$ implies

$$\phi(z - Cq) - qe^{-\gamma t}\eta(z) \leq w(t, z) \leq \kappa, \quad z \in \mathbb{R}, \quad t \geq -h.$$

Proof. By Remark 2.2, the statements of Lema 2.1 will not change if we replace $\phi(z)$ with a shifted profile $\phi(z + b)$, $b \in \mathbb{R}$. Taking $b = -Cq$, we complete the proof of Corollary 2.3. \blacksquare

As an immediate consequence of Lemma 2.1 and Corollary 2.3, we obtain the stability of the wavefront solution $u(t, x) = \phi(x + ct)$ with respect to the norm $|\cdot|_\lambda$:

Corollary 2.4 *For every $\epsilon > 0$ there exists $\delta > 0$ such that $|\phi(\cdot + s_0) - w(s, \cdot)|_\lambda < \delta$, $s \in [-h, 0]$, implies that $|\phi(\cdot + s_0) - w(t, \cdot)|_\lambda < \epsilon$ for all $t \geq 0$.*

Proof. Without loss of generality, we can assume that $s_0 = 0$. From Theorem 1.4 and Proposition 2 from [42], we know that $\phi'(z) = O(e^{\lambda_2 z})$ at $-\infty$. This implies that $|\phi'(z)| \leq K \min\{1, e^{\lambda_2 z}\}, z \in \mathbb{R}$, for some positive K . In this way, for each fixed $p \in \mathbb{R}$,

$$0 < \phi'(z + p) \leq K \min\{1, e^{\lambda_2(z+p)}\} \leq K e^{\lambda_2|p|} \min\{1, e^{\lambda_2 z}\}, z \in \mathbb{R}.$$

Fix $\epsilon > 0$ and consider $\delta \in (0, q_0^+) \cap (0, \epsilon/(1 + K_1))$, where $K_1 = C K e^{\lambda_2 C q_0^+}$. Next, assume that $|\phi(\cdot) - w(s, \cdot)|_\lambda < \delta, s \in [-h, 0]$. This yields that

$$\phi(z) - \delta\eta(z) < w(s, z) < \phi(z) + \delta\eta(z), s \in [-h, 0], z \in \mathbb{R},$$

and therefore, due to Lemma 2.1 and Corollary 2.3,

$$\phi(z - C\delta) - \delta\eta(z) < w(t, z) < \phi(z + C\delta) + \delta\eta(z), t \geq 0, z \in \mathbb{R}.$$

Now, for some $\hat{s} \in (0, C\delta)$, it holds

$$\begin{aligned} \phi(z + C\delta) &= \phi(z) + \phi(z + C\delta) - \phi(z) = \phi(z) + C\delta\phi'(z + \hat{s}) \\ &\leq \phi(z) + C K e^{\lambda_2 C q_0^+} \delta \min\{1, e^{\lambda_2 z}\} \leq \phi(z) + K_1 \delta\eta(z). \end{aligned} \quad (19)$$

After establishing a similar lower bound for $\phi(z - C\delta)$, we get

$$\phi(z) - (K_1 + 1)\delta\eta(z) < w(t, z) < \phi(z) + (K_1 + 1)\delta\eta(z), t \geq 0, z \in \mathbb{R},$$

that is, $|\phi(\cdot) - w(t, \cdot)|_\lambda < \delta(K_1 + 1) = \epsilon, t \geq 0$. ■

In addition, Lemma 2.1 yields the following useful result

Corollary 2.5 *Assume that $w_0(s, x)$ satisfies (IC). Then there exist positive γ, ζ_1 such that*

$$\phi(z - \zeta_1) - \sigma e^{-\gamma t} \eta(z) \leq w(t, z) \leq \phi(z + \zeta_1) + q_0^+ e^{-\gamma t} \eta(z + \zeta_1), z \in \mathbb{R}, t \geq -h. \quad (20)$$

Proof. First, we will show that inequality (12) holds for $w_0(s, z - \zeta_0)$ if we take sufficiently large ζ_0 . Indeed, let z' be such that $\phi(z') + q_0^+ \eta(z') = \kappa$ and define ζ_0 from

$$A e^{-\mu \zeta_0} = q_0^+ \min\{e^{-\mu z'}, e^{(\lambda - \mu)z'}\}.$$

Then, for all $z \geq z', s \in [-h, 0]$, it holds that $w(s, z - \zeta_0) \leq 1 \leq \phi(z) + q_0^+ \eta(z)$. Furthermore, because of the assumption (IC2) and the inequality $\lambda < \mu$, we have, for all $z \leq z', s \in [-h, 0]$, that $w(s, z - \zeta_0) \leq A e^{\mu(z - \zeta_0)} =$

$$q_0^+ \min\{e^{-\mu z'}, e^{(\lambda - \mu)z'}\} e^{\mu z} \leq q_0^+ \min\{e^{\mu(z - z')}, e^{\lambda z}\} \leq q_0^+ \eta(z) < \phi(z) + q_0^+ \eta(z).$$

Therefore, due to (13),

$$0 \leq w(t, z - \zeta_0) \leq \phi(z + C q_0^+) + q_0^+ e^{-\gamma t} \eta(z), z \in \mathbb{R}, t \geq -h.$$

Hence, setting $\zeta_1 = \zeta_0 + C q_0^+$ and using the translation invariance of equation (1), we obtain the second inequality in (20).

Similarly, there exists z'' such that

$$\begin{aligned}\phi(z + z'') - q_0^- \eta(z) &\leq 0 \leq w(s, z + B) \leq \kappa, \quad z \leq 0, \quad s \in [-h, 0]; \\ \phi(z + z'') - q_0^- \eta(z) &\leq \kappa - \sigma \leq w(s, z + B) \leq \kappa, \quad z \geq 0, \quad s \in [-h, 0].\end{aligned}$$

Hence, by (15) and Remark 2.2, we obtain

$$\phi(z + z'' - Cq_0^-) - q_0^- e^{-\gamma t} \eta(z) \leq w(s, z + B) \leq \kappa, \quad z \in \mathbb{R}, \quad t \geq -h.$$

As a consequence, the both inequalities in (20) hold if we take $\zeta_1 = \zeta_0 + C(q_0^+ + q_0^-) + |B - z''|$. \blacksquare

Remark 2.6 Observe that the hypothesis (IC3) was not used to prove the right-hand side inequality in (20).

Next, it should be noted that the variable shift $\epsilon(t)$ in $w_+(t, z)$ was needed only to assure the inequality $\mathcal{N}w_+(t, z) \geq 0$ on the finite interval $z - ch + \epsilon(t) \in [z_1, z_2]$, cf. Step III. This observation suggests the following important modification of Lemma 2.1 (where we will take the same constants $\delta, \gamma > 0$ which were defined in Step II of the proof of Lemma 2.1):

Lemma 2.7 Let $w(t, z)$ be a solution of (10), (11) with $\tilde{w}_0(s, z) \in [0, \kappa]$. Take $\delta > 0$ as in (17) and let $R > ch$ be such that

$$\begin{aligned}0 \leq w(t, z), \quad \phi(z) \leq \delta, \quad \text{if } z \leq -R + ch, \quad t \geq -h, \quad \text{and} \\ |w(t, z) - \kappa|, \quad |\phi(z) - \kappa| < \delta \quad \text{if } z \geq R - ch, \quad t \geq -h.\end{aligned}$$

Furthermore, suppose that $w(s, z) \leq \phi(z) + \delta \eta(z)$ for all $(s, z) \in [-h, 0] \times \mathbb{R}$ and $w(t, z) \leq \phi(z)$ for all $(t, z) \in \mathbb{R}_+ \times [-R - ch, R + ch]$. Then $w(t, z) \leq \phi(z) + \delta \eta(z) e^{-\gamma t}$ for all $z \in \mathbb{R}, t \geq 0$.

Proof. Set $\rho(t, z) = w(t, z) - \phi(z)$, then, for some $\xi(t, z)$ lying between points $w(t - h, z - ch)$ and $\phi(z - ch)$,

$$\rho_t(t, z) = \rho_{zz}(t, z) - c\rho_z(t, z) - \rho(t, z) + g'(\xi(t, z))\rho(t - h, z - ch), \quad z \in \mathbb{R}, \quad t \geq 0.$$

Since $\xi(t, z) \in [0, \delta]$ for $z \leq -R$, $t \geq 0$, and $\kappa - \xi(t, z) \in [0, \delta]$ for $z \geq R$, $t \geq 0$, we find that $r(t, z) := \delta \eta(z) e^{-\gamma t}$ satisfies

$$\begin{aligned}r_t(t, z) - r_{zz}(t, z) + cr_z(t, z) + r(t, z) - g'(\xi(z, t))r(t - h, z - ch) = \\ \delta e^{-\gamma t} ((1 - \gamma)\eta(z) - \eta''(z) + c\eta'(z) - e^{\gamma h} g'(\xi(t, z))\eta(z - ch)) > 0, \quad |z| \geq R, \quad t \geq 0.\end{aligned}$$

In addition, by our assumptions, the piece-wise smooth function $\delta(t, z) := w(t, z) - (\phi(z) + r(t, z))$ satisfies the inequalities $\delta(t, \pm R) \leq 0$, $|\delta(t, z)| \leq 2\kappa + \delta$, $t \geq 0$, $z \in \mathbb{R}$; $\delta(s, z) \leq 0$, $s \in [-h, 0]$, $z \in \mathbb{R}$. In consequence,

$$\delta_{zz}(t, z) - \delta_t(t, z) - c\delta_z(t, z) - \delta(t, z) > -g'(\xi(t, z))\delta(t - h, z - ch) \geq 0,$$

for all $t \in [0, h]$, $|z| \geq R$. By the Phragmén-Lindelöf principle [31], we conclude that $\delta(t, z) \leq 0$ for all $t \in [0, h]$, $|z| \geq R$. Since we also have assumed that $w(t, z) \leq \phi(z)$ for all $(t, z) \in \mathbb{R}_+ \times [-R - ch, R + ch]$, we obtain that $\delta(t, z) \leq 0$ for all $t \in [0, h]$, $z \in \mathbb{R}$.

Finally, repeating the above arguments on the intervals $[h, 2h]$, $[2h, 3h]$, \dots , we complete the proof of the lemma. \blacksquare

Finally, before starting with the proof of Theorems 1.3 and 1.4, we will establish the following compactness result.

Lemma 2.8 *Assume that continuous function $w : [-h, +\infty) \times \mathbb{R} \rightarrow [0, \kappa]$ is a classical solution, for $t > 0$, of equation (10) and that $t_j \rightarrow +\infty$. Then there exists a subsequence $\{t_{j_k}\} \subset \{t_j\}$ such that $w(t_j + s, z)$ converges, uniformly on each rectangle $[-h, 0] \times [-m, m]$, $m \in \mathbb{N}$, to the restriction $w_*(s, z)$, $(s, z) \in [-h, 0] \times \mathbb{R}$, of some entire solution $w_* : \mathbb{R}^2 \rightarrow [0, \kappa]$ of equation (10).*

Proof. First, we observe that, for each fixed $t > h$, function $g(w(t - h, z - ch))$ is locally Lipschitz continuous in $z \in \mathbb{R}$ and therefore w, w_z, w_{zz} are Hölder continuous in $(h, +\infty) \times \mathbb{R}$, cf. [26, Theorem 1]. Next, fix an arbitrary positive $T > 2h + 2$ and $m \in \mathbb{N}$ and consider, for $t_j > T + 2h$, solutions $w_j(t, z) = w(t_j + t, z)$, $(t, z) \in D_+ := [-T, T] \times [-m - 1, m + 1 + ch]$, of the equation

$$w_t(t, z) = w_{zz}(t, z) - cw_z - w(t, z) + g_j(t, z),$$

where $g_j(t, z) := g(w_j(t - h, z - ch))$. We claim that, for each $\alpha \in (0, 1)$, there exists a positive K depending only on m, T, α such that the Hölder norms

$$|g_j|_\alpha^D = \sup_{(t,z) \in D} |g_j(t, z)| + \sup_{(t,z) \neq (s,x) \in D} \frac{|g_j(t, z) - g_j(s, x)|}{(|x - z|^2 + |t - s|)^{\alpha/2}}$$

are uniformly bounded in $D := [-T + 1 + h, T] \times [-m, m]$ by K (i.e. $|g_j|_\alpha^D \leq K$ for all j . Observe that $|g_j|_\alpha^{D_+}$ is finite due to [26, Theorem 1]). In fact, since g satisfies the Lipschitz condition on $[0, \kappa]$, it suffices to establish the uniform boundedness of $|w_j|_\alpha^{D_1}$ in a bigger domain $D_1 := [-T + 1, T] \times [-m, m + ch] \subset D_+$. Obviously, w_j solves in D_+ the initial-boundary value problem $w = w_j|_{\partial D_+}$ where $w_j|_{\partial D_+}$ denotes the restriction of w_j on the parabolic boundary $\partial D_+ := \{-T\} \times [-m - 1, m + 1 + ch] \cup [-T, T] \times \{-m - 1, m + 1 + ch\}$ of D_+ . Let $\rho : [-T, T] \rightarrow [0, 1]$ be some nondecreasing smooth function such that $\rho([-T, -T + 0.25]) = 0$, $\rho([-T + 0.75, T]) = 1$. Then $w_j = w_{j,1} + w_{j,2}$ where $w_{j,1}$ is the solution of the initial-boundary value problem $w = 0|_{\partial D_+}$ for the equation

$$w_t(t, z) = w_{zz}(t, z) - cw_z - w(t, z) + \rho(t)g_j(t, z),$$

and $w_{j,2}$ solves the initial-boundary value problem $w = w_j|_{\partial D_+}$ for the equation

$$w_t(t, z) = w_{zz}(t, z) - cw_z - w(t, z) + (1 - \rho(t))g_j(t, z).$$

Next, since $|g_j(t, z)| \leq \kappa$ for all $(t, z) \in D_+$, $j \in \mathbb{N}$, a priori estimate (of the type $1 + \delta$) established in [11, Theorem 4, Chapter 7] guarantees that $|w_{j,1}|_\alpha^{D_+} \leq K_1$, $j \in \mathbb{N}$, where K_1 depends only on m, T, α . As consequence, since $\sup_{D_+} |w_j|$, $j \in \mathbb{N}$, are uniformly bounded by κ , we deduce that $\sup_{D_+} |w_{j,2}| = \sup_{D_+} |w_j - w_{j,1}|$, $j \in \mathbb{N}$, are also uniformy bounded. In addition, $(1 - \rho(t))g_j(t, z) = 0$ in $[-T + 0.75, T] \times [-m - 1, m + 1 + ch]$, so

that we can invoke the interior Schauder estimates (see, e.g., [11, Theorem 5, Chapter 3]) in order to deduce that $|w_{j,1}|_\alpha^{D_1} \leq K_2, j \in \mathbb{N}$, where $K_2 > 0$ depends only on α and K_1 . Hence, $|w_j|_\alpha^{D_1} \leq K_1 + K_2, j \in \mathbb{N}$, and therefore $|g_j|_\alpha^D \leq K := L_g(K_1 + K_2)$ for all j .

Applying now Theorem 15 from [11, Chapter 3], we conclude that there exists a subsequence $\{t_{j_k}\} \subset \{t_j\}$ such that $w_{j_k}(t, z)$ converges, uniformly on $[-T+2+h, T-1] \times [-m+1, m-1]$, to the classical solution $w_{T,m} : [-T+2+2h, T] \times [-m+1, m-1] \rightarrow [0, \kappa]$ of equation (10). Finally, considering $m, T \rightarrow +\infty$ and applying a standard diagonal argument, we can assume that $w_{j_k}(t, z)$ converges, uniformly on compact subsets of \mathbb{R}^2 to an entire classical solution $w_* : \mathbb{R}^2 \rightarrow [0, \kappa]$ of the functional differential equation (10). Observe that the arguments used to estimate $|w_j|_\alpha^{D_1}$ can be also applied without changes to w_* so that $|w_*|_\alpha^{D_1} \leq K_1 + K_2$ with the same K_1, K_2 . \blacksquare

Remark 2.9 Due to Lemma 2.8, we can define ω -limit set $\omega(w_0)$ which consists from the restrictions $w_*(s, z), (s, z) \in [-h, 0] \times \mathbb{R}$, of all possible entire limit solutions $w_* = \lim_{k \rightarrow +\infty} w_{j_k}$ to (10) (which are obtained by considering all possible sequences $\{t_j\}$ converging to $+\infty$ in this lemma). Since each w_* is an entire solution, the set $\omega(w_0)$ is invariant. Furthermore, since $|w_*|_\alpha^D \leq K_1 + K_2$ where K_1, K_2 depend only on D and α , the set $\omega(w_0)$ is pre-compact with respect to the topology of the uniform convergence on bounded subsets of $[-h, 0] \times \mathbb{R}$. Actually $\omega(w_0)$ is compact in the mentioned topology since each element of $\omega(w_0)$ can uniformly (on bounded sets) be approximated by w_j .

Theorem 2.10 Assume that $u = \phi(x + c_*t)$, $c_* > c_\#$, is a pushed traveling front to equation (1). If initial function w_0 satisfies all conditions (IC) then, for some $z_0 \in \mathbb{R}$, the classical solution $w = w(t, z)$ of the initial value problem (10), (11) asymptotically converges to a shifted front profile:

$$\lim_{t \rightarrow \infty} |w(t, \cdot) - \phi(\cdot + z_0)|_\lambda = 0. \quad (21)$$

In order to prove the above theorem, instead of looking for an appropriate Lyapunov functional (as it was done in [10, 33]) for functional differential equation (10), we will use the Berestycki and Nirenberg method of the sliding solutions as well as some ideas of the approach developed by Ogiwara and Matano in [30].

Proof. By Corollary 2.5, Lemma 2.8 and Remark 2.9, solution $w = w(t, z)$ of the initial value problem (10), (11) has a compact invariant ω -limit set $\omega(w_0)$ such that for some fixed ζ_1 , it holds

$$\phi(z - \zeta_1) \leq w_*(0, z) \leq \phi(z + \zeta_1), \quad z \in \mathbb{R}, \quad \text{for each } w_* \in \omega(w_0). \quad (22)$$

Then the set

$$A = \{a \in \mathbb{R} : w_*(0, z) \leq \phi(z + a), \quad z \in \mathbb{R}, \quad \text{for each } w_* \in \omega(w_0)\}$$

contains ζ_1 and has $-\zeta_1$ as its lower bound. Therefore $\hat{a} = \inf A$ is a well defined finite number. Due to continuity of ϕ , we have that $\hat{a} \in A$ so that

$$w_*(0, z) \leq \phi(z + \hat{a}), \quad z \in \mathbb{R}, \quad \text{for each } w_* \in \omega(w_0).$$

In fact, since $\omega(w_0)$ is an invariant set, we have that $w_*(t, z) \leq \phi(z + \hat{a})$, $z \in \mathbb{R}$, $t \in \mathbb{R}$. Suppose now for a moment that $w_*(0, z') = \phi(z' + \hat{a})$ for some finite z' and some $w_* \in \omega(w_0)$. Therefore, since g is an increasing function, the strong maximum principle yields $w_*(t, z) \equiv \phi(z + \hat{a})$ for all $t \leq 0$, $z \in \mathbb{R}$. In particular, $w_*(0, z) \equiv \phi(z + \hat{a})$ so that, for some sequence $t_n \rightarrow +\infty$, it holds that $w(t_n + s, z) \rightarrow \phi(z + \hat{a})$ uniformly with respect to $s \in [-h, 0]$ and z from compact subsets of \mathbb{R} . In addition, Corollary 2.5 allows to evaluate the difference $|w(t_n + s, z) - \phi(z + \hat{a})|/\eta(z)$ in some fixed neighbourhood of the endpoints $z = -\infty$ and $z = +\infty$ and to conclude that $w(t_n + s, z) \rightarrow \phi(z + \hat{a})$, $n \rightarrow +\infty$, in the norm $|\cdot|_\lambda$ and uniformly with respect to $s \in [-h, 0]$. By Corollary 2.4, the latter convergence implies (21) with $z_0 = \hat{a}$ that completes the proof of the theorem in the case when $w_*(0, z') = \phi(z' + \hat{a})$ holds for some finite z' .

In this way, we are left to consider the situation when

$$w_*(0, z) < \phi(z + \hat{a}), \quad z \in \mathbb{R}, \quad \text{for each } w_* \in \omega(w_0). \quad (23)$$

In virtue of (22), for any given $\delta > 0$, we can find $R > 3ch + 1$ sufficiently large to have, for all $w_* \in \omega(w_0)$,

$$w_*(0, z) < \phi(z + \hat{a}) < \delta, \quad \text{for } z \leq -R + ch + 1, \quad \phi(z + \hat{a}) > w_*(0, z) > \kappa - \delta, \quad \text{for } z \geq R - ch - 1.$$

Then, using (23) and the compactness of the set

$$\{w_*(0, \cdot) : [-R + ch + 1, R - ch - 1] \rightarrow [0, \kappa], \quad w_* \in \omega(w_0)\} \subset C[-R + ch + 1, R - ch - 1],$$

we deduce the existence of $\varsigma \in (0, 1)$ such that

$$w_*(0, z) < \phi(z + \hat{a} - \varsigma), \quad z \in [-R + ch + 1, R - ch - 1], \quad \text{for each } w_* \in \omega(w_0).$$

It is clear that

$$\phi(z + \hat{a}) < \kappa < \phi(z + \hat{a} - \varsigma) + \delta, \quad z \geq R - ch.$$

Without the loss of generality, we also can suppose that $\varsigma \in (0, 1)$ is such that

$$\phi(z + \hat{a}) < \phi(z + \hat{a} - \varsigma) + \delta e^{\lambda z}, \quad z \leq -R + ch + 1.$$

Indeed, observe that $\phi'(z) \leq C e^{\lambda_2 z}$, $z \leq 0$, and therefore, for some $\xi \in (z + \hat{a} - \varsigma, z + \hat{a})$,

$$\phi(z + \hat{a}) - \phi(z + \hat{a} - \varsigma) = \phi'(\xi)\varsigma \leq C e^{\lambda_2(z + \hat{a})}\varsigma \leq \delta e^{\lambda z}, \quad z \leq 0,$$

once $\varsigma \leq e^{-\lambda_2 \hat{a}} \delta / C$. Hence, invoking again the invariance property of $\omega(w_0)$, we can conclude that for each $w_* \in \omega(w_0)$ it holds

$$w_*(t, z) \leq \phi(z + \hat{a} - \varsigma) + \delta \eta(z), \quad t \in \mathbb{R}, \quad z \in \mathbb{R},$$

and

$$w_*(t, z) \leq \phi(z + \hat{a} - \varsigma), \quad z \in [-R + ch + 1, R - ch - 1], \quad t \in \mathbb{R}.$$

By Lemma 2.7, this yields $w_*(t, z) \leq \phi(z + \hat{a} - \varsigma) + \delta \eta(z) e^{-\gamma t}$, $t \geq 0$, $z \in \mathbb{R}$, where $\hat{a}, \varsigma, \gamma$ do not depend on the particular choice of $w_* \in \omega(w_0)$. In consequence, since $w_*(t, z)$ is an entire solution, we obtain that actually $w_*(0, z) \leq \phi(z + \hat{a} - \varsigma)$, $z \in \mathbb{R}$, for all $w_* \in \omega(w_0)$. This contradicts to the definition of \hat{a} and shows that the case (23) can not happen. \blacksquare

3. Proof of Proposition 1.2

First, observe that for each g satisfying the assumptions of Proposition 1.2, we can find a *monotone* function $g_1 : [0, \kappa] \rightarrow [0, \kappa]$ possessing all the properties of g and such that $g_1(x) \leq g(x)$. Therefore, in view of the comparison principle, it will not restrict the generality if we will assume additionally the monotonicity of g .

Here, we follow an approach, proposed by Aronson and Weinberger in [2, Theorem 3.1], and based on the maximum principle. In the mentioned work, it was established, for every $\epsilon \in (0, \kappa)$ and appropriate $b_\epsilon > 0$, the existence of a positive solution $q = q(x) \leq \epsilon$ to the Dirichlet boundary value problem

$$\begin{aligned} q''(x) - q(x) + g(q(x)) &= 0, \quad x \in I_\epsilon := (0, b_\epsilon), \\ q(0) &= q(b_\epsilon) = 0. \end{aligned}$$

Since we are interested in the asymptotic behavior of $u(t, x) \geq 0$ and $w_0(s, x) \not\equiv 0$, without loss of generality, due to the strong maximum principle we can suppose that $w_0(s, x) > 0$ for all $(s, x) \in [-h, 0] \times \mathbb{R}$. But then we can choose $\epsilon > 0$ small enough to have $q(x) \leq u(x, s)$ for all $x \in I_\epsilon$, $s \in [-h, 0]$. Let χ_A denote the characteristic function of subset $A \subset \mathbb{R}$. Consider solution $u = u_\epsilon(t, x)$ of the initial value problem $u_\epsilon(s, x) = \chi_{I_\epsilon}(x)q(x)$, $s \in [-h, 0]$, $x \in \mathbb{R}$, to equation (1). Then the difference $\delta(t, x) = q(x) - u_\epsilon(t, x)$ satisfies $\delta(t, 0) \leq 0$, $\delta(t, b_\epsilon) \leq 0$, $t \geq 0$, and

$$\delta_t(t, x) - \delta_{xx}(t, x) + \delta(t, x) = g(q(x)) - g(u_\epsilon(t, x)) \leq 0, \quad (t, x) \in [0, h] \times I_\epsilon.$$

Hence, by the maximum principle, $u_\epsilon(t, x) \geq q(x)$ on $[0, h] \times I_\epsilon$. Repeating the same argument on $[h, 2h] \times I_\epsilon$, we obtain that $u_\epsilon(t, x) \geq q(x)$ for all $(t, x) \in [h, 2h] \times I_\epsilon$. It is clear that this procedure yields the inequality $q(x) \leq u_\epsilon(t, x) < 1$ in $[0, +\infty) \times I_\epsilon$. But then, since for each positive l , it holds that $\chi_{I_\epsilon}(x)q(x) = u_\epsilon(s, x) \leq u_\epsilon(s + l, x)$, $s \in [-h, 0]$, $x \in \mathbb{R}$, we can use the Phragmén-Lindelöf principle, in order to conclude that $u_\epsilon(t + l, x) \geq u_\epsilon(t, x)$ for all $(t, x) \in [0, h] \times \mathbb{R}$. Similarly to the above analysis, step by step, we can extend the latter inequality for all $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$. Thus, for each fixed $x \in \mathbb{R}$, $u_\epsilon(t, x)$ is a non-decreasing bounded function of $t \geq 0$. Let $u_\epsilon(x) = \lim_{t \rightarrow +\infty} u_\epsilon(t, x)$, then $u_\epsilon(x) \in (0, \kappa]$ for every $x \in \mathbb{R}$.

Now, a direct application of Lemma 2.8 shows that $u_\epsilon(x)$ solves

$$u''(x) - u(x) + g(u(x)) = 0, \quad x \in \mathbb{R}$$

while the convergence $u_\epsilon(x) = \lim_{t \rightarrow +\infty} u_\epsilon(t, x)$ is uniform on compact subsets of \mathbb{R} . Since $g(u) - u > 0$ on $(0, \kappa)$, the function $u_\epsilon(x)$ cannot take (local) minimal values in $(0, \kappa)$. This implies the existence of $u_\epsilon(\pm\infty) \in \{0, \kappa\}$. In other words, $u(x)$ is a positive stationary traveling wave solution of equation (1) considered with $h = 0$. It is well known [15] that this is possible only when $u_\epsilon(x) \equiv \kappa$.

Finally, we complete the proof by observing that, due to the maximum principle, it holds $u_\epsilon(t, x) \leq u(t, x)$ on $[0, \infty) \times \mathbb{R}$.

In Sections 4 and 5, we are always assuming that all the conditions of Theorem 1.5 are satisfied (recall also that, by simplifying the notation, we write c instead of c_*). The proof of this theorem will follow from a series of lemmas. In the first of them we improve the asymptotic relation $u(t, 0) = \kappa + o(1)$ at $+\infty$ known from Proposition 1.2. As we show below, this convergence is actually of the exponential type.

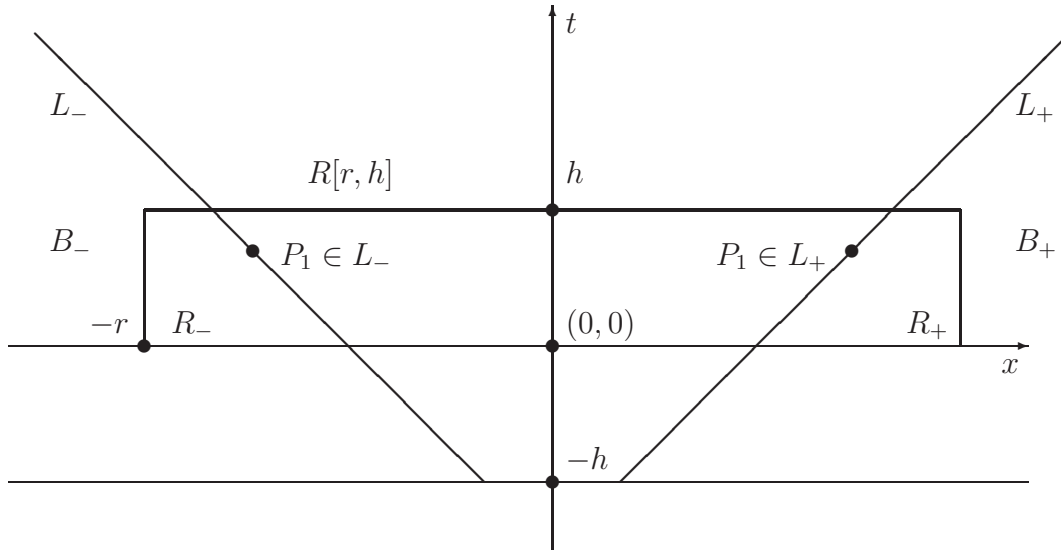


Figure 1. Domains $R[r, h], R_1, R_{\pm} \subset B_{\pm}$ and lines L_{\pm} .

$$u(t, 0) \geq \kappa - qe^{-\nu t} \quad \text{for all } t \geq 0. \quad (24)$$
$$-\lambda^2 + c\lambda + 1 - \gamma - g'(\bar{s})e^{-\lambda ch + \gamma h} > 0 \text{ for } \bar{s} < \delta,$$

where $\delta < \delta_1^*, \gamma_1^*, z_1 < 0 < z_2$ are defined in Steps I, II of Lemma 2.1. Following [10], we will construct a sub-solution to (1) of the form

$$u_-(t, x) = \phi_+(t, x) + \phi_-(t, x) - \kappa - q(t, x),$$

where $\phi_{\pm}(t, x) = \phi(\pm x + ct - \epsilon(t))$, $q(t, x) = \gamma e^{-\gamma t} \theta(t, x)$ with $\theta(t, x) \leq 1$ are defined by

$$\theta(t, x) := \eta(-|x| + ct - \epsilon(\infty) - z_1) = \begin{cases} e^{\lambda(-x+ct-\epsilon(\infty)-z_1)}, & \text{if } (t, x) \in B_+, \\ e^{\lambda(x+ct-\epsilon(\infty)-z_1)}, & \text{if } (t, x) \in B_-, \\ 1, & \text{if } (t, x) \in [-h, \infty) \times \mathbb{R} \setminus (B_+ \cup B_-), \end{cases}$$

$$B_{\pm} := [-h, \infty) \times \mathbb{R} \cap \{(t, x) : \mp x + ct - \epsilon(\infty) < z_1\},$$

$$L_{\pm} := [-h, \infty) \times \mathbb{R} \cap \{(t, x) : \mp x + ct - \epsilon(\infty) = z_1\},$$

with an appropriate $\epsilon(t)$ satisfying $\epsilon'(t) > 0$, $\epsilon(t) < 0$. Then $\epsilon(\infty) + z_1 < -ch$ and therefore $B_+ \cap B_- = \emptyset$. See also Figure 1. Set

$$\mathcal{N}_1 u_-(t, x) := (u_-)_t(t, x) - (u_-)_{xx}(t, x) + u_-(t, x) - g(u_-(t - h, x)),$$

$$\tilde{\phi}_\pm(t, x) := \phi(\pm x + c(t - h) - \epsilon(t)) < \phi_\pm(t - h, x).$$

Since $u_-(t, x) = u_-(t, -x)$, it holds that $\mathcal{N}_1 u_-(t, x) = \mathcal{N}_1 u_-(t, -x)$. In view of monotonicity of g and ϕ , we have

$$\begin{aligned} \mathcal{N}_1 u_-(t, x) &\leq g(\tilde{\phi}_+(t, x)) + g(\tilde{\phi}_-(t, x)) - g(\tilde{\phi}_+(t, x) + \tilde{\phi}_-(t, x) - \kappa - q(t - h, x)) \\ &\quad - \epsilon'(t)[\phi'(x + ct - \epsilon(t)) + \phi'(-x + ct - \epsilon(t))] - \kappa - q(t, x) + q_{xx}(t, x) - q_t(t, x). \end{aligned}$$

Claim I: $\mathcal{N}_1 u_-(t, x) = \mathcal{N}_1 u_-(t, -x) < 0$ for $x \geq 0$, $t > 0$, $(t, x) \notin L_\pm$.

By Step II of Lemma 2.1, $-x + c(t - h) - \epsilon(t) \geq z_2$ implies $\kappa - \delta/2 < \tilde{\phi}_-(t, x)$. Since $x \geq 0$, we also have $\kappa - \delta/2 < \tilde{\phi}_-(t, x) \leq \tilde{\phi}_+(t, x)$. By Step I of Lemma 2.1, for $\gamma \in (0, \gamma_1^*)$, $\gamma < \sigma - \delta/2$,

$$\begin{aligned} &g(\tilde{\phi}_-(t, x)) - g(\tilde{\phi}_-(t, x) - [\kappa - \tilde{\phi}_+(t, x) + q(t - h, x)]) \\ &\leq e^{-\gamma h}(1 - 2\gamma)[\kappa - \tilde{\phi}_+(t, x) + q(t - h, x)]. \end{aligned}$$

Hence, since $\theta(t, x)$ is non-decreasing in t , we have, for $t > 0$, that $\mathcal{N}_1 u_-(t, x) \leq$

$$\begin{aligned} &(1 - 2\gamma)e^{-\gamma h}[(\kappa - \tilde{\phi}_+(t, x)) + q(t - h, x)] + g(\tilde{\phi}_+(t, x)) - \kappa - q(t, x) + q_{xx}(t, x) - q_t(t, x) \\ &\leq -\tilde{\phi}_+(t, x) + (1 - 2\gamma)\gamma e^{-\gamma t}\theta(t - h, x) + g(\tilde{\phi}_+(t, x)) - q(t, x) + q_{xx}(t, x) - q_t(t, x) \\ &\leq g(\tilde{\phi}_+(t, x)) - \tilde{\phi}_+(t, x) + q(t, x) \begin{cases} \lambda^2 - c\lambda - 1 + \gamma + (1 - 2\gamma)e^{-\lambda ch}, & \text{if } (t, x) \in B_+, \\ -2\gamma, & \text{if } (t, x) \in [0, \infty) \times \mathbb{R}^+ \setminus B_+. \end{cases} \end{aligned}$$

On the other hand, it is known (see e.g. [42, Remark 1]) that, for some $C > 0$, it holds

$$0 \leq \kappa - \tilde{\phi}_+(t, x) \leq Ce^{-\lambda_3 \epsilon(t)} e^{\lambda_3(x+ct)}, \quad t \geq -h, \quad x \in \mathbb{R}. \quad (25)$$

This implies that, for $t > 0, x \geq 0$, $-x + c(t - h) - \epsilon(t) \geq z_2$, it holds that

$$\begin{aligned} \mathcal{N}_1 u_-(t, x) &\leq Ce^{-\lambda_3 \epsilon(\infty)} e^{\lambda_3 x} e^{\lambda_3 ct} \\ &\quad + \gamma e^{-\gamma t} \begin{cases} e^{\lambda(-x+ct-\epsilon(\infty)-z_1)}[\lambda^2 - c\lambda - 1 + \gamma + (1 - 2\gamma)e^{-\lambda ch}], & \text{if } (t, x) \in B_+, \\ -2\gamma, & \text{if } (t, x) \in [0, \infty) \times \mathbb{R}^+ \setminus B_+, \end{cases} \\ &\leq e^{-\gamma t} \begin{cases} e^{-\lambda x}[\gamma(\lambda^2 - c\lambda - 1 + \gamma + (1 - 2\gamma)e^{-\lambda ch}) + Ce^{-\lambda_3 \epsilon(\infty)}], & \text{if } (t, x) \in B_+, \\ -2\gamma^2 + Ce^{-\lambda_3 \epsilon(\infty)}, & \text{if } (t, x) \in [0, \infty) \times \mathbb{R}^+ \setminus B_+. \end{cases} \end{aligned}$$

As a consequence, there exists large negative $\epsilon(\infty)$ (depending on γ and λ_3) such that

$$\mathcal{N}_1 u_-(t, x) < 0 \text{ for } t > 0, \quad -x + c(t - h) - \epsilon(t) \geq z_2, \quad (t, x) \notin L_+.$$

Next, if $-x + c(t - h) - \epsilon(t) \leq z_1$ then $0 \leq \tilde{\phi}_-(t, x) \leq \delta/2$ and $(t, x) \in B_+$. Thus $\theta(t - h, x) = e^{\lambda(-x+ct-ch-\epsilon(\infty)-z_1)}$ and, for some $\bar{s} < \delta/2$,

$$g(\tilde{\phi}_-(t, x)) - g(\tilde{\phi}_-(t, x) - [\kappa - \tilde{\phi}_+(t, x) + q(t - h, x)]) = g'(\bar{s})[\kappa - \tilde{\phi}_+(t, x) + q(t - h, x)].$$

Thus, recalling that $z_1 < 0$, for large $\epsilon(\infty) < 0$ (which depends on γ and λ_3), we get

$$\begin{aligned} \mathcal{N}_1 u_-(t, x) &\leq g'(\bar{s})[\kappa - \tilde{\phi}_+(t, x) + q(t - h, x)] - q(t, x) + q_{xx}(t, x) - q_t(t, x) \\ &\leq \gamma e^{-\gamma t} e^{\lambda(-x+ct-\epsilon(\infty)-z_1)} [\lambda^2 - c\lambda - 1 + \gamma + g'(\bar{s})e^{(\gamma-c\lambda)h}] + g'(\bar{s})[\kappa - \tilde{\phi}_+(t, x)] \\ &\leq \gamma e^{-\gamma t} e^{-\lambda x} [\lambda^2 - c\lambda - 1 + \gamma + g'(\bar{s})e^{(\gamma-c\lambda)h}] + g'(\bar{s})C e^{\lambda_3(x+ct-\epsilon(t))} \\ &\leq e^{-\lambda x} e^{-\gamma t} \{ \gamma [\lambda^2 - c\lambda - 1 + \gamma + g'(\bar{s})e^{(\gamma-c\lambda)h}] + g'(\bar{s})C e^{-\lambda_3\epsilon(\infty)} \} < 0. \end{aligned}$$

Finally, consider $z_1 \leq -x + c(t - h) - \epsilon(t) \leq z_2$. Recall that $\beta > 0$ defined in Step II of Lemma 2.1 depends only on δ, ϕ and satisfies $\beta < \min_{\zeta \in [z_1, z_2+ch]} \phi'(\zeta)$. Therefore, if we take $\epsilon'(t) = \alpha \gamma e^{-\gamma t}$ for some $\alpha > 0$, then

$$|g(\tilde{\phi}_-(t, x)) - g(\tilde{\phi}_+(t, x) + \tilde{\phi}_-(t, x) - \kappa - q(t - h, x))| \leq L_g [C e^{-\lambda_3\epsilon(t)} e^{\lambda_3(x+ct)} + q(t - h, x)].$$

In consequence, if α is sufficiently large then $\mathcal{N}_1 u_-(x, t) \leq C L_g e^{\lambda_3(-\epsilon(t)+x+ct)}$

$$\begin{aligned} &+ \begin{cases} \gamma e^{-\gamma t} \{-\alpha\beta + e^{\lambda(-x+ct-\epsilon(\infty)-z_1)} [\lambda^2 - c\lambda - 1 + \gamma + L_g e^{(\gamma-\lambda c)h}]\}, & \text{if } (t, x) \in B_+, \\ \gamma e^{-\gamma t} [-\alpha\beta + \gamma - 1 + L_g e^{\gamma h}], & \text{if } (t, x) \in [0, \infty) \times \mathbb{R}^+ \setminus B_+, \end{cases} \leq \\ &e^{-\gamma t} \{ \gamma [-\alpha\beta + L_g e^{\gamma h}] + C L_g e^{-\lambda_3\epsilon(\infty)} e^{\lambda_3 x} \} < 0, \text{ for } (t, x) \in \mathbb{R}^+ \times \mathbb{R}^+. \end{aligned}$$

Claim II: There exists $t_0 > 0$ such that $u_-(s, x) \leq u(s + t_0, x)$ for $x \in \mathbb{R}$, $s \in [-h, 0]$.

Since $\lambda_2 > \lambda$, there exists $r_0 > 0$ depending on $\epsilon(-h), \epsilon(\infty), z_1$ such that, for $s \in [-h, 0]$,

$$u_-(s, x) \leq \phi(-|x| + cs - \epsilon(s)) - \gamma \eta(-|x| + cs - z_1 - \epsilon(\infty)) < 0 \text{ if } |x| \geq r_0.$$

Clearly, $u_-(s, x) < \kappa$ for all $|x| \leq r_0, s \in [-h, 0]$ and therefore, by Proposition 1.2, $u_-(s, x) < u(t_0 + s, x)$, $|x| \leq r_0, s \in [-h, 0]$, for an appropriate $t_0 > 0$.

Claims I and II allow to complete the proof of Lemma 4.1. First, for $r > 0$, consider rectangle $R[r, h] = [0, h] \times [-r, r]$. Set $\delta(t, x) := u_-(t, x) - u(t + t_0, x)$, the function $\delta(t, x)$ is smooth in $[-h, +\infty) \times \mathbb{R} \setminus \{L_- \cup L_+\}$ (in particular, in the regions $R_\pm = R[r, h] \cap B_\pm$, $R_1 = R[r, h] \setminus (\bar{R}_+ \cup \bar{R}_-)$). Since $\delta(s, x) \leq 0$ in $[-h, 0] \times \mathbb{R}$ and

$$\delta_t(t, x) - \delta_{xx}(t, x) + \delta(t, x) \leq g(u_-(t - h, x)) - g(u(t + t_0 - h, x)) \leq 0,$$

for all $(t, x) \in [0, h] \times \mathbb{R} \setminus \{L_- \cup L_+\}$, the maximum principle assures that the function $\delta(t, x)$ in $R[r, h]$ is either negative or it reaches a non-negative maximum at a point $P_1 = (t_1, x_1)$ belonging to $\partial R_1 \cup \partial R_+ \cup \partial R_- \setminus \{h\} \times (-r, r)$. It is easy to see that $P_1 \notin L_\pm$. Indeed, if $P_1 \in L_\pm$ (see Fig. 1) then $\delta_x(P_1+) - \delta_x(P_1-) = \gamma \lambda e^{-\gamma t_1} > 0$. Thus the non-negative maximum of $\delta(t, x)$ on $R[r, h]$ is attained at a point from the parabolic boundary of $R[r, h]$. In consequence, the usual maximum principle holds for each $R[r, h]$ so that, just as it was done in Step V of the proof of Lemma 12, we can appeal to the Phragmén-Lindelöf principle in order to conclude that $\delta(t, z) \leq 0$ for all $t \in [0, h]$, $z \in \mathbb{R}$. Applying the above argument consecutively on the intervals $[h, 2h]$, $[2h, 3h]$, ... we find that $\delta(t, x) \leq 0$ for all $t \geq -h, x \in \mathbb{R}$. Therefore, in view of (25),

$$u(t + t_0, 0) \geq 2\phi(ct - \epsilon(\infty)) - \kappa - \gamma e^{-\gamma t} \geq \kappa - q' e^{-\gamma t}, \quad t \geq -h,$$

for some sufficiently large $q' > \gamma$. Obviously, this yields (24) with appropriate $q > q'$. ■

Corollary 4.2 *The conclusion of Lemma 4.1 holds without the assumption $\lambda_1 < -\lambda_3$.*

Proof. First, we observe that there exists a monotone function $\hat{g}(x) \leq g(x)$ satisfying the hypothesis **(H)** and such that the equation

$$u_t(t, x) = u_{xx}(t, x) - u(t, x) + \hat{g}(u(t - h, x)) \quad (26)$$

has a pushed wavefront $\hat{\phi}(\hat{c}t + x)$ with the associated eigenvalues $\hat{\lambda}_1, \hat{\lambda}_3 = \lambda_3$ such that $\hat{\lambda}_1 < -\hat{\lambda}_3$. Indeed, let $g_n(x) \leq g(x)$, be a sequence of monotone functions satisfying **(H)**, coinciding with $g(x)$ on $[1/n, \kappa]$, uniformly on $[0, \kappa]$ converging to $g(x)$ and such that $\lim_{n \rightarrow +\infty} g'_n(0) = 1$. Then [23, Lemma 3.5] implies that $c_n := c_*(g_n) \leq c := c_*(g)$ while the proof of Proposition 1.1 shows that $\liminf_{n \rightarrow +\infty} c_n \geq c$. This means that $\lim_{n \rightarrow +\infty} c_n = c > c_\# > c_\#^{(n)}$ and $\lim_{n \rightarrow +\infty} \lambda_1^{(n)} = 0 < -\lambda_3$ where, similarly to $c_\#, \lambda_1$, the numbers $c_\#^{(n)}, \lambda_1^{(n)}$ are determined from the characteristic equation (4) with $g'(0)$ replaced by $g'_n(0)$.

In consequence, if $\hat{u}(t, x)$ denotes the solution of the initial value problem (2) for (26), with $w_0 \not\equiv 0$, then Lemma 4.1 implies that $\hat{u}(t, 0) \geq \kappa - qe^{-\nu t}$, $t > 0$, for some positive q, ν . Finally, by comparing initial value problems (1), (2) and (26), (2) and invoking the Phragmén-Lindelöf principle, we get that $u(t, 0) \geq \hat{u}(t, 0) \geq \kappa - qe^{-\nu t}$ for all $t > 0$. ■

Corollary 4.3 *Assume that all the conditions of Theorem 1.5 are satisfied. Then there exist $K > 1, t_1 > 0$ and $z', z'' \in \mathbb{R}$ such that*

$$u(t, x) \geq \phi(-|x| + ct - z') - Ke^{-\gamma t} \eta(-|x| + ct - z'') \text{ for all } t > t_1 - h, x \in \mathbb{R}.$$

Proof. Consider $u_-(t, x) = \phi(-|x| + ct - \epsilon(t)) - \gamma e^{-\gamma t} \theta(t, x)$. Analysing the proof of Claim I of Lemma 4.1, we can easily find that it is also valid for $x \neq 0$ if we replace $\phi_+(t, x)$ with κ . Moreover, in such a case, the restriction $\lambda_1 < -\lambda_3$ is unnecessary (recall that this restriction appears due to the term $\phi_+(t, x) - \kappa$). Hence, we conclude that, for an appropriate choice of $\epsilon(t)$, it holds $\mathcal{N}_1 u_-(t, x) \leq 0$ for all $x > 0, t > 0, (t, x) \notin L_+$. Since $\mathcal{N}_1 u_-(t, x) = \mathcal{N}_1 u_-(t, -x)$, we conclude that u_- is a sub-solution in the region $x \neq 0, t > 0, (t, x) \notin L_\pm$. In addition, for some sufficiently large $t_1 > 0$, it holds

$$\begin{aligned} u(t + t_1, 0) &\geq \kappa - qe^{-\gamma(t+t_1)} > \phi(ct - \epsilon(-h)) - \gamma e^{-\gamma t} \eta(ct - \epsilon(\infty) - z_1) = \\ &\phi(ct - \epsilon(-h)) - \gamma e^{-\gamma t} > u_-(t, 0) \text{ for all } t \geq -h. \end{aligned}$$

Now, arguing as in Claim II of the proof of Lemma 4.1, we can also assume that t_1 is chosen in such a way that $u_-(s, x) \leq u(s + t_1, x)$ for $x \in \mathbb{R}, s \in [-h, 0]$. But then, using the Phragmén-Lindelöf principle in the regions $[hj, h(j+1)] \times [0, +\infty), [hj, h(j+1)] \times (-\infty, 0], j = 0, 1, \dots$ according to the procedure established in the last paragraph of the proof of Lemma 4.1, we conclude that, for all $x \in \mathbb{R}, t \geq t_1 - h$, it holds that

$$\begin{aligned} u(t, x) &\geq u_-(t - t_1, x) = \phi(-|x| + c(t - t_1) - \epsilon(t - t_1)) - \gamma e^{-\gamma(t-t_1)t} \theta(t - t_1, x) \geq \\ &\phi(-|x| + c(t - t_1) - \epsilon(\infty)) - \gamma e^{-\gamma(t-t_1)t} \eta(-|x| + c(t - t_1) - \epsilon(\infty) - z_1). \end{aligned}$$

This completes the proof of the corollary. ■

Lemma 4.4 Assume all the conditions of Theorem 1.5 and suppose that for some sequence $t_n \rightarrow +\infty$ and $s_1, s_2 \in \mathbb{R}$, it holds

$$\lim_{n \rightarrow \infty} \sup_{x \leq 0} |u(t_n + s, x) - \phi(x + c(t_n + s) + s_1)| / \eta(x + ct_n) = 0, \quad (27)$$

$$\lim_{n \rightarrow \infty} \sup_{x \geq 0} |u(t_n + s, x) - \phi(-x + c(t_n + s) + s_2)| / \eta(-x + ct_n) = 0, \quad (28)$$

uniformly on $s \in [-h, 0]$. Then for every $\delta > 0$ there exists $T(\delta) > 0$ such that

$$\begin{aligned} \sup_{x \leq 0} \frac{|u(t, x) - \phi(x + ct + s_1)|}{\eta(x + ct)} &< \delta \quad \text{for all } t \geq T(\delta), \\ \sup_{x \geq 0} \frac{|u(t, x) - \phi(-x + ct + s_2)|}{\eta(-x + ct)} &< \delta \quad \text{for all } t \geq T(\delta). \end{aligned} \quad (29)$$

Proof. It suffices to establish (29), since $u(t, -x)$ also solves equation (1) and satisfies all the hypotheses of Theorem 1.5. Without restricting generality, we can take $s_1 = 0$. We know from Corollary 4.2 that $u(t, 0) \geq \kappa - qe^{-\nu t}$, $t \geq 0$. Fix $\gamma \in (0, \min\{\nu, -c\lambda_3\})$ and consider $\epsilon(t) = \alpha\delta\gamma^{-1}e^{-\gamma t}$ (with α defined in Step II of Lemma 2.1) and

$$u_n(t, x) = \phi(x + ct + ct_n - \alpha\gamma^{-1}e^{\gamma h}\delta + \epsilon(t)) - \delta e^{-\gamma t}\eta(x + ct + ct_n).$$

Let positive integer $N = N(\delta)$ be such that $\delta e^{\nu t_N} > q$ and

$$\sup_{(s, x) \in [-h, 0] \times (-\infty, 0]} \frac{|u(t_n + s, x) - \phi(x + c(t_n + s))|}{\eta(x + c(t_n + s))} < \delta \quad \text{for all } n \geq N(\delta).$$

Then we obtain, for all for $(s, x) \in [-h, 0] \times (-\infty, 0]$,

$$u_N(s, x) \leq \phi(x + c(t_N + s)) - \delta\eta(x + c(t_N + s)) \leq u(t_N + s, x).$$

Let us show now that a similar relation holds for all $(t, x) \in [t_N, \infty) \times \{0\}$ once $N(\delta)$ is large. Indeed, we have that $u_N(t, 0) \leq \kappa - \delta e^{-\gamma t}$ for all $t \geq 0$ so that,

$$u(t + t_N, 0) - u_N(t, 0) \geq \delta e^{-\gamma t} - qe^{-\nu t_N}e^{-\nu t} > 0, \quad t \geq 0.$$

Next, observe that $u_n(t, x) = w_-(t, x + c(t + t_n))$ where w_- is defined in Lemma 2.1 (by Remark 2.2, the summand $-\alpha\gamma^{-1}e^{\gamma h}$ within the argument of ϕ doesn't matter). Since $\delta < \sigma$, we find that $(u_n)_t(t, x) - (u_n)_{xx}(t, x) + u_n(t, x) - g(u_n(t - h, x)) = (\mathcal{N}w_-)(t, x + c(t + t_n)) < 0$ for all $(t, x) \in [0, \infty) \times \mathbb{R}$, $x + ct + ct_n \neq 0$. Furthermore, if $x' + ct' + ct_n = 0$ at some point (x', t') then $(u_n)_x(t', x' + 0) - (u_n)_x(t', x' - 0) = \lambda\delta e^{-\gamma t'} > 0$. Therefore, repeatedly applying the Phragmén-Lindelöf principle in the regions $[hj, h(j + 1)] \times (-\infty, 0]$, $j = 0, 1, \dots$ according to the procedure established in the last paragraph of the proof of Lemma 4.1, we conclude that, for all $x \leq 0$, $t \geq -h$,

$$u(t + t_N, x) \geq u_N(t, x) \geq \phi(x + c(t + t_N) - \alpha\gamma^{-1}e^{\gamma h}\delta) - \delta\eta(x + c(t + t_N)).$$

Hence, taking positive constant $K_1 = K_1(\alpha, \gamma, h)$ as in (19), we obtain that

$$u(t, x) \geq \phi(x + ct) - \delta(1 + K_1)\eta(x + ct), \quad t \geq t_N - h, \quad x \leq 0. \quad (30)$$

On the other hand, by our assumptions, for all for $(s, x) \in [-h, 0] \times (-\infty, 0]$,

$$u(t_N + s, x) \leq \phi(x + c(t_N + s)) + \delta\eta(x + c(t_N + s)). \quad (31)$$

If, in addition, $N = N(\delta)$ is so large that

$$\phi(c(t_N + s)) + \delta\eta(c(t_N + s)) > \kappa, \quad s \in [-h, 0],$$

then (31) holds also for all $(s, x) \in [-h, 0] \times \mathbb{R}$. Therefore for $\delta \in (0, q_0]$, by Lemma 2.1,

$$u(t + t_N, x) \leq \phi(x + c(t_N + t) + C\delta) + \delta e^{-\gamma t} \eta(x + c(t_N + t)), \quad t \geq 0, \quad x \in \mathbb{R},$$

for positive $C > 0$ defined in Lemma 2.1. Next, due to (19), for all $(t, x) \in \mathbb{R}^2$, we have

$$\phi(x + c(t_N + t) + C\delta) \leq \phi(x + c(t_N + t)) + K_1 \delta \eta(x + c(t_N + t)).$$

In consequence, we obtain

$$u(t, x) \leq \phi(x + ct) + \delta(1 + K_1) \eta(x + ct), \quad \text{for } t \geq t_N, \quad x \leq 0.$$

The latter inequality together with (30) imply (29). \blacksquare

5. Proof of Theorem 1.5: main arguments

Set $z = x + ct$ and $w(t, z) := u(t, x) = u(t, z - ct)$, then $w(t, z)$ satisfies equation (10), (11) for $(t, z) \in \mathbb{R}_+ \times \mathbb{R}$ and possesses a compact and invariant ω -limit set $\omega(w_0)$ defined in Remark 2.9. Consider the semi-infinite strip $\Omega = \{(s, z) \in [-h, 0] \times \mathbb{R}, \quad z \leq -ch\}$. By Corollary 2.5 and Remark 2.6, for some $K > 0$, $\zeta_1 \in \mathbb{R}$, it holds

$$w(t, z) \leq \phi(z + \zeta_1) + K e^{-\gamma t} \eta(z + \zeta_1), \quad z \in \mathbb{R}, \quad t \geq -h. \quad (32)$$

Therefore the set

$$A = \{a \in \mathbb{R} : v(s, z) \leq \phi(z + a), \quad (s, z) \in \Omega, \quad \text{for each } v \in \omega(w_0)\}$$

is non-empty. Since, by Corollary 4.3,

$$\phi(z - z_1) - K \gamma e^{-\gamma t} \eta(z - z_1) \leq w(t, z), \quad z \leq ct, \quad t \geq -h, \quad (33)$$

A is bounded below. Set $\hat{a} := \inf A$, obviously, $\hat{a} \in A$. We claim that $v_*(s_*, z_*) = \phi(z_* + \hat{a})$ for some $(s_*, z_*) \in \Omega$ and $v_* \in \omega(w_0)$. Indeed, suppose on the contrary that

$$v(s, z) < \phi(z + \hat{a}) \quad \text{for all } (s, z) \in \Omega, \quad v \in \omega(w_0). \quad (34)$$

For positive ς and an entire solution $v \in \omega(w_0)$, $v : \mathbb{R}^2 \rightarrow [0, \kappa]$, consider $\rho(t, z) = v(t, z) - \phi(z + \hat{a} - \varsigma)$. Let $R > ch$ be such that $\phi(-R + \zeta_1) < \delta$. Then, for each $\xi(t, z)$ lying between points $v(t - h, z - ch)$ and $\phi(z + \hat{a} - \varsigma - ch)$ with $z \leq -R$, $t \in \mathbb{R}$, we have $\xi(t, z) \in (0, \delta)$. Next, set $r(t, z) = \eta(z) e^{-\gamma t}$ and let δ be as in (17). In view of (17),

$$\begin{aligned} r_t(t, z) - r_{zz}(t, z) + c r_z(t, z) + r(t, z) &= \eta(z) e^{-\gamma t} [1 - \gamma - \lambda^2 + c\lambda] \\ &\geq \eta(z) e^{-\gamma t} g'(\xi(t, z)) e^{-\lambda ch + \gamma h}, \quad t > 0, \quad z \leq -R, \quad \varsigma > 0, \quad v \in \omega(w_0), \end{aligned}$$

$$\rho_t(t, z) = \rho_{zz}(t, z) - c \rho_z(t, z) - \rho(t, z) + g'(\xi(t, z)) \rho(t - h, z - ch), \quad t \in \mathbb{R}, \quad z \leq -R.$$

On the other hand, since the set $\omega(w_0)$ is compact and invariant (the latter means that $\omega(w_0)$ consists of *entire* solutions $v : \mathbb{R}^2 \rightarrow [0, \kappa]$) and ϕ increases on \mathbb{R} , we can fix $\varsigma > 0$ such that (34) implies

$$v(t, z) < \phi(z + \hat{a} - \varsigma), \quad t > 0, \quad -R \leq z \leq -ch, \quad v \in \omega(w_0). \quad (35)$$

Without loss of generality, we can also suppose that ς is sufficiently small to meet

$$\phi(z + \hat{a}) < \phi(z + \hat{a} - \varsigma) + \eta(z)e^{-\gamma s} \quad \text{for all } z \in \mathbb{R}, \quad s \in [-h, 0]. \quad (36)$$

Now, we set $\delta(t, z) := \rho(t, z) - r(t, z)$. Note that, by (34) and (36), for all for $s \in [-h, 0]$, $z \leq -ch$, it holds

$$\delta(s, z) = v(s, z) - (\phi(z + \hat{a} - \varsigma) + \eta(z)e^{-\gamma s}) < v(s, z) - \phi(z + \hat{a}) < 0,$$

and therefore, in virtue of the above mentioned properties of ρ, r ,

$$\begin{aligned} \delta_{zz}(t, z) - \delta_t(t, z) - c\delta_z(t, z) - \delta(t, z) &\geq -g'(\xi(t, z))\rho(t - h, z - ch) \\ &+ \eta(z)e^{-\gamma t}g'(\xi(t, z))e^{-\lambda ch + \gamma h} = -g'(\xi(t, z))\delta(t - h, z - ch) > 0 \quad \text{for } z \leq -R, \quad t \in [0, h]. \end{aligned}$$

Taking into account that, due to (35), it holds $-\kappa - 1 < \delta(t, z) < 0$ for all $t \in [0, h]$, $-R \leq z \leq -ch$, we can invoke now the Phragmén-Lindelöf principle [31] in order to conclude that $\delta(t, z) < 0$ for all $t \in [0, h]$, $z \leq -R$. But then, by repeating the above argument for the time intervals $[h, 2h]$, $[2h, 3h]$, \dots , and using (35) we conclude that

$$v(t, z) \leq \phi(z + \hat{a} - \varsigma) + \eta(z)e^{-\gamma t}$$

for all $t \geq 0$, $z \leq -ch$. Due to the invariance property of $\omega(w_0)$ this yields

$$v(s, z) < \phi(z + \hat{a} - \varsigma), \quad -h \leq s \leq 0, \quad z \leq -ch, \quad v \in \omega(w_0),$$

contradicting the definition of \hat{a} .

Hence, $w_*(s_*, z_*) = \phi(z_* + \hat{a})$ for some $(s_*, z_*) \in \Omega$ and $w_* \in \omega(w_0)$. Therefore, by the strong principle maximum and invariance property of $\omega(w_0)$, we obtain that $\phi \in \omega(w_0)$.

Next, it follows from (32) and (33) that, for all $z \leq ct$, $t \geq -h$, it holds

$$|w(t, z) - \phi(z + \hat{a})| \leq \phi(z + \zeta_1) - \phi(z - z_1) + Ke^{-\gamma t}(\eta(z - z_1) + \eta(z + \zeta_1)).$$

In consequence, for each $\epsilon > 0$ we can find $T(\epsilon) > 0$ such that

$$|w(t + s, z) - \phi(z + \hat{a})| < \epsilon \quad \text{for } t \geq T(\epsilon), \quad cT(\epsilon) \leq z \leq ct, \quad s \in [-h, 0].$$

and

$$\frac{|w(t, z) - \phi(z + \hat{a})|}{\eta(z)} < \epsilon \quad \text{for } t \geq T(\epsilon), \quad z \leq -cT(\epsilon), \quad s \in [-h, 0].$$

On the other hand, since $\phi \in \omega(w_0)$, there exist $t_n \rightarrow \infty$ and an integer $n(\epsilon)$ so that:

$$\frac{|w(t_n + s, z) - \phi(z + \hat{a})|}{\eta(-cM)} < \epsilon, \quad n \geq n(\epsilon), \quad |z| \leq cT(\epsilon), \quad s \in [-h, 0].$$

Obviously, the last three inequalities imply (27). Moreover, by considering the solution $\hat{u}(t, x) = u(t, -x)$ together with the obtained sequence $\{t_n\}$, we can see that (28) is also satisfied for a subsequence $\{t_{n_j}\} \subset \{t_n\}$ and an appropriate s_2 . Finally, an application of Lemma 4.4 completes the proof of Theorem 1.5.

Acknowledgments

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